Journal of Advances in Mathematics and Computer Science

Volume 39, Issue 7, Page 55-69, 2024; Article no.JAMCS.119485

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

On the Number of Cyclic Codes Over $\mathbb{Z}_{31}$

John Joseph O. Ondiany [a*], Obogi Robert Karieko [a], Lao Hussein Mude [b] and Fred Nyamitago Monari [a]

[a] *Department of Mathematics and Actuarial Sciences, Kisii University, P. O. Box 1957-40200, Kisii, Kenya.*
[b] *Department of Pure and Applied Sciences, Kirinyaga University, P. O. Box 143-10300, Kerugoya, Kenya.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: https://doi.org/10.9734/jamcs/2024/v39i71912

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/119485

Received: 27/04/2024
Accepted: 02/07/2024
Published: 10/07/2024

Original Research Article

Abstract

Let n be a positive integer, $y^n - 1$ cyclotomic polynomial and $\mathbb{Z}_q$ be a given finite field. In this study we determined the number of cyclic codes over $\mathbb{Z}_{31}$. First, we partitioned the cyclotomic polynomial $y^n - 1$ using cyclotomic cosets 31 mod n and factorized $y^n - 1$ using case to case basis. Each monic divisor obtained is a generator polynomial and generate cyclic codes. The results obtained are useful in the field of coding theory and more especially, in error correcting codes.

Keywords: Code; cyclic codes; cyclotomic coset.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

**Corresponding author: E-mail: jondiany5@gmail.com;*

Cite as: *Ondiany, John Joseph O., Obogi Robert Karieko, Lao Hussein Mude, and Fred Nyamitago Monari. 2024. "On the Number of Cyclic Codes Over $\mathbb{Z}_{31}$". Journal of Advances in Mathematics and Computer Science 39 (7):55-69. https://doi.org/10.9734/jamcs/2024/v39i71912.*

1 Introduction

The study of cyclic codes have attracted attention of many researchers since the advent of cryptography, this is because cyclic codes have direct applications in coding theory in particular error correction coding [1, 2, 3]. The problem of finding codes which are optimal, that will transmit a wide varieties of messages fast and correct many errors has been a motivating factor in the study of codes [4, 5, 6, 7]. Factorization of polynomials over a finite field is of interest in algebraic coding theory, computational number theory, computer algebra, and cryptography among other areas see [8, 9, 10, 11, 12]. Cyclotomic cosets has been used to partition Cyclotomic polynomials $y^n - 1$ [13, 14, 15]. Cyclic codes of length n over finite fields $\mathbb{Z}_q$ for which $q \leq 23$ has been fully characterized and a general formula for computing the number of cyclic codes given see [16, 17, 18, 4, 19, 20, 21, 22]. In this study the number of cyclic codes over $\mathbb{Z}_{31}$ are determined and generalizations made over $\mathbb{Z}_{31}$.

1.1 Definitions

i. **Code:** Let F be a finite set with q symbols, there are q^n different sequences of length n, of these only q^k are codewords since the r check digits within any codeword are completely determined by the k message digits. The set consisting of q^k codewords of length n is called a code.

ii. **Cyclic Code:** Let C be a lnear code over a finite field $GF(q)$ of block n, C is called a cyclic code, if for every codeword $a_0, a_1, a_2, ..., a_n$ from C, the word $a_n, a_0, a_1, a_2, ..., a_{n-1}$ in C obtain by a cyclic right shift of component is also a codeword. This also involves the left shift. Therefore a linear code C is cyclic precisely when it is invariant under all cyclic shifts.

iii. **Cyclotomic Coset:** Let n be relatively to q. The cyclotomic coset of q mod n is defined by $C_i = \{i.q^j \bmod n \in \mathbb{Z}_n, j = 0, 1, 2, 3, ...\}$

2 Main Results

2.1 Number of cyclotomic cosets and factorization of $y^n - 1$ over $\mathbb{Z}_{31}$

i. Let $n = 1$, Then $C_i = \{i.31^j \bmod 1 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$ We get 1 cyclotomic cosets 31 mod 1. We factorize $y - 1$. $y - 1 = (y + 30)$. The number of irreducible monic polynomial is 1 of degree 1.

ii. Let $n = 2$, Then $C_i = \{i.31^j \bmod 2 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1\}$. We get 2 cyclotomic cosets 31 mod 2. We factorize $y^2 - 1$. $y^2 - 1 = (y - 1)(y + 1) = (y + 30)(y + 1)$. The number of irreducible monic polynomial are also 2 of degree 1.

iii. Let $n = 3$, Then $C_i = \{i.31^j \bmod 3 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$. We get 3 cyclotomic cosets 31 mod 3. We factorize $y^3 - 1$. $y^3 - 1 = (y - 1)(y^2 + y + 1) = (y + 30)(y + 26)(y + 6)$. The number of irreducible monic polynomial are also 3 of degree 1.

iv. Let $n = 4$, Then $C_i = \{i.31^j \bmod 4 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 3\}$, $C_2 = \{2\}$. We get 3 cyclotomic cosets 31 mod 4. We factorize $y^4 - 1$. $y^4 - 1 = (y^2 - 1)(y^2 + 1) = (y - 1)(y+)(y^2 + 1)$. The number of irreducible monic polynomial are also 3, 2 of degree 1 and 1 of degree 2.

v. Let $n = 5$, Then $C_i = \{i.31^j \bmod 5 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$,$C_3 = \{3\}$, $C_4 = \{4\}$. We get 5 cyclotomic cosets 31 mod 5. We factorize $y^5 - 1$. $y^5 - 1 = (y - 1)(y^4 + y^3 + y^2 + y + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)$. The number of irreducible monic polynomial are also 5 of degree 1.

vi. Let $n = 6$, Then $C_i = \{i.31^j \bmod 6 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$,$C_3 = \{3\}$, $C_4 = \{4\}$, $C_5 = \{5\}$. We get 6 cyclotomic cosets 31 mod 6. We factorize $y^6 - 1$. $y^6 - 1 = (y^3 - 1)(y^3 + 1) = (y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25)$. The number of irreducible monic polynomial are also 6 of degree 1.

vii. Let $n = 7$, Then $C_i = \{i.31^j \bmod 7 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 2, 3, 4, 5, 6\}$. We get 2 cyclotomic cosets 31 mod 7. We factorize $y^7 - 1$. $y^7 - 1 = (y+30)(y^6+y^5+y^4+y^3+y^4+y^3+y^2+y+1)$. The number of irreducible monic polynomial are also 2, 1 of degree 1 and 1 of degree 7.

vii. Let $n = 8$, Then $C_i = \{i.31^j \bmod 8 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 7\}$, $C_2 = \{2, 6\}$, $C_3 = \{3, 5\}$,$C_4 = \{4\}$. We get 5 cyclotomic cosets 31 mod 8. We factorize y^8-1 . $y^8-1 = (y^4-1)(y^4+1) = (y^2-1)(y^2+1)(y^4+1) = (y+30)(y+1)(y^2+1)(y^4+1) = (y+30)(y+1)(y^2+1)(y^2+8y+1)(y^2+23y+1)$. The number of irreducible monic polynomial are also 5, 2 of degree 1 and 3 of degree 2.

ix. Let $n = 9$, Then $C_i = \{i.31^j \bmod 9 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 4, 7\}$, $C_2 = \{2, 6, 8\}$, $C_3 = \{3\}$, $C_5 = \{5\}$. We get 5 cyclotomic cosets 31 mod 9. We factorize $y^9 - 1$. $y^9 - 1 = (y-1)(y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1) = (y+30)(y+26)(y+6)(y^3+1)(y^3+26)$. The number of irreducible monic polynomial are also 5, 3 of degree 1 and 2 of degree 3.

x. Let $n = 10$, Then $C_i = \{i.31^j \bmod 10 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$, $C_5 = \{5\}$, $C_6 = \{6\}$, $C_7 = \{7\}$, $C_8 = \{8\}$, $C_9 = \{9\}$. We get 10 cyclotomic cosets 31 mod 10. We factorize $y^{10} - 1$. $y^{10} - 1 = (y^5 - 1)(y^5 + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)(y^5 + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)(y + 1)(y + 2)(y + 4)(y + 8)(y + 16)$. The number of irreducible monic polynomial are also 10 of degree 1.

xi. Let $n = 11$, Then $C_i = \{i.31^j \bmod 11 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 9, 4, 3, 5\}$, $C_2 = \{2, 6, 7, 8, 10\}$. We get 3 cyclotomic cosets 31 mod 11. We factorize $y^{11} - 1$. $y^{11} - 1 = (y-1)(y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1) = (y+30)(y^5+10y^4+30y^3+y^2+9y+30)(y^5+22y^4+30y^3+y^2+21y+30)$. The number of irreducible monic polynomial are also 3,1 of degree 1 and two of degree 5.

xii. Let $n = 12$, Then $C_i = \{i.31^j \bmod 12 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 7\}$, $C_2 = \{2\}$, $C_3 = \{3, 9\}$, $C_4 = \{4\}$,
$C_5 = \{5, 11\}$,$C_6 = \{6\}$, $C_8 = \{8\}$,$C_{10} = \{10\}$. We get 9 cyclotomic cosets 31 mod 12. We factorize $y^{12}-1$. $y^{12}-1 = (y^6-1)(y^6+1) = (y^3-1)(y^3+1)(y^6+1) = (y+30)(y+26)(y+6)(y+1)(y+5)(y+25)(y^2+1)(y^2+5)(y^2+25)$ The number of irreducible monic polynomial are also 9, 6 of degree 1 and 3 of degree 2.

xiii. Let $n = 13$, Then $C_i = \{i.31^j \bmod 13 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 5, 8, 12\}$, $C_2 = \{2, 10, 11, 13\}$, $C_4 = \{4, 7, 9, 6\}$. There are 4 cyclotomic cosets 31 mod 13. We factorize $y^{13}-1$. $y^{13}-1 = (y-1)(y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1) = (y+30)(y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1) = (y+30)(y^4+4y^3+11y^2+4y+)(y^4+6y^3+29y^2+6y+1)(y^4+22y^3+27y^2+22y+1)$. The number of irreducible monic polynomial are also 4, 1 of degree 1 and 3 of degree 4.

xiv. Let $n = 14$, Then $C_i = \{i.31^j \bmod 14 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 3, 9, 13, 11, 5\}$, $C_2 = \{2, 4, 6, 8, 10, 12\}$, $C_7 = \{7\}$ 4 cyclotomic cosets 31 mod 14 are obtained. We factorize $y^{14} - 1$. $y^{14}-1 = (y^7-1)(y^7+1) = (y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y^7+1) = (y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y+1)(y^6+30y^5+y^4+30y^3+y^2+30y+1)$. The number of irreducible monic polynomial are also 4, 2 of degree 1 and 2 of degree 6.

xv. Let $n = 15$, Then $C_i = \{i.31^j \bmod 15 \in Z_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$, $C_5 = \{5\}$, $C_6 = \{6\}$, $C_7 = \{7\}$, $C_8 = \{8\}$, $C_9 = \{9\}$, $C_{10} = \{10\}$, $C_{11} = \{11\}$, $C_{12} = \{12\}$, $C_{13} = \{13\}$, $C_{14} = \{14\}$. There are 15 cyclotomic cosets 31 mod 15. We factorize $y^{15} - 1$. $y^{15} - 1 = (y - 1)(y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y+30)(y+26)(y+6)(y+3)(y+21)(y+29)(y+23)(y+27)(y+17)(y+11)(y+24)(y+22)(y+12)(y+13)(y+15)$. The number of irreducible monic polynomial are also 15 of degree 1.

xvi. Let $n = 16$, Then $C_i = \{i.31^j \bmod 16 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 15\}$, $C_2 = \{2, 14\}$, $C_3 = \{3, 13\}$, $C_4 = \{4, 12\}$, $C_5 = \{5, 11\}$, $C_6 = \{6, 10\}$, $C_8 = \{8\}$, $C_9 = \{9, 7\}$. There are 9 cyclotomic cosets 31 mod 16. We factorize $y^{16} - 1$. $y^{16} - 1 = (y^8 - 1)(y^8 + 1) = (y^2 - 1)(y^2 + 1)(y^4 + 1)(y^8 + 1) = (y+30)(y+1)(y^2+1)(y^2+8y+1)(y^2+23y+1)(y^8+1) = (y+30)(y+1)(y^2+1)(y^2+8y+1)(y^2+26y+1)(y^2+23y+1)(y^2+5y+1)(y^2+14y+1)(y^2+17y+1)$. The number of irreducible monic polynomial are also 9, 2 of degree 1 and 7 of degree 2.

xvii. Let $n = 17$, Then $C_i = \{i.31^j \text{ mod } 17 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ There are 2 cyclotomic cosets 31 mod 17. We factorize $y^{17} - 1$. $y^{17} - 1 = (y - 1)(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)$. The number of irreducible monic polynomial are also 2, 1 of degree 1 and 1 of degree 16.

xviii. Let $n = 18$, Then $C_i = \{i.31^j \text{ mod } 18 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 7, 13\}$, $C_2 = \{2, 8, 14\}$, $C_3 = \{3\}$, $C_4 = \{4, 10, 16\}$, $C_5 = \{5, 11, 17\}$, $C_6 = \{6\}$, $C_9 = \{9\}$, $C_{12} = \{12\}$, $C_{15} = \{15\}$. There are 10 cyclotomic cosets 31 mod 18. We factorize $y^{18} - 1$. $y^{18} - 1 = (y^9 - 1)(y^9 + 1) = (y + 30)(y + 26)(y + 6)(y^3 + 6)(y^3 + 26)(y^9 + 1) = (y + 30)(y + 26)(y + 6)(y^3 + 6)(y^3 + 26)(y + 1)(y + 5)(y + 25)(y^3 + 5)(y^3 + 25)$. The number of irreducible monic polynomial are also 10, 6 of degree 1 and 4 of degree 3.

xix. Let $n = 19$, Then $C_i = \{i.31^j \text{ mod } 19 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 8, 10, 11, 12, 18\}$, $C_2 = \{2, 3, 5, 14, 16, 17\}$, $C_4 = \{4, 6, 7, 9, 13, 15\}$. There are 4 cyclotomic cosets 31 mod 19. We factorize $y^{19} - 1$. $y^{19} - 1 = (y - 1)(y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y^6 + 4y^5 + 14y^4 + 18y^3 + 14y^2 + 4y + 1)(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)(y^6 + 25y^5 + 5y^4 + y^3 + 25y^3 + 5y + 1)$. The number of irreducible monic polynomial are also 4, 1 of degree 1 and 3 of degree 6.

xx. Let $n = 20$, Then $C_i = \{i.31^j \text{ mod } 20 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 11\}$, $C_2 = \{2\}$, $C_3 = \{3, 13\}$, $C_4 = \{4\}$, $C_5 = \{5, 15\}$, $C_6 = \{6\}$, $C_7 = \{7, 17\}$, $C_8 = \{8\}$, $C_9 = \{9, 19\}$, $C_{10} = \{10\}$, $C_{12} = \{12\}$, $C_{14} = \{14\}$, $C_{16} = \{16\}$, $C_{18} = \{18\}$. There are 15 cyclotomic cosets 31 mod 20.
We factorize $y^{20} - 1$. $y^{20} - 1 = (y^{10} - 1)(y^{10} + 1) = (y^5 - 1)(y^5 + y)(y^{10} + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)(y + 1)(y + 2)(y + 4)(y + 8)(y + 16)(y^{10} + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)(y + 1)(y + 2)(y + 4)(y + 8)(y + 16)(y^2 + 1)(y^2 + 2)(y^2 + 4)(y^2 + 8)(y^2 + 16)$. The number of irreducible monic polynomial are also 15, 10 of degree 1 and 5 of degree 2.

xxi. Let $n = 21$, Then $C_i = \{i.31^j \text{ mod } 21 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 4, 10, 13, 16, 19\}$, $C_2 = \{2, 5, 8, 11, 17, 20\}$, $C_3 = \{3, 6, 9, 12, 15, 18\}$, $C_7 = \{7\}$, $C_{14} = \{14\}$. There are 6 cyclotomic cosets 31 mod 21.
We factorize $y^{21} - 1$. $y^{21} - 1 = (y - 1)(y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y + 6)(y + 26)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^{12} + 30y^{11} + y^{10} + 30y^9 + y^8 + 30y^7 + y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1) = (y + 30)(y + 6)(y + 26)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)(y^6 + 25y^5 + 5y^4 + y^3 + 25y^2 + 5y + 1)$ The number of irreducible monic polynomial are also 6, 3 of degree 1 and 3 of degree 6.

xxii. Let $n = 22$, Then $C_i = \{i.31^j \text{ mod } 22 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 3, 5, 9, 15\}$, $C_2 = \{2, 6, 8, 10, 18\}$, $C_4 = \{4, 14, 12, 16, 20\}$, $C_7 = \{7, 13, 17, 19, 21\}$, $C_{11} = \{11\}$. There are 6 cyclotomic cosets 31 mod 22. We factorize $y^{22} - 1$. $y^{22} - 1 = (y^{11} - 1)(y^{11} + 1) = (y + 30)(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)(y + 1)(y^{10} + 30y^9 + y^8 + 30y^7 + y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1) = (y + 30)(y + 1)(y^5 + 9y^4 + 30y^3 + 30y^2 + 21y + 1)(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)(y^5 + 21y^4 + 30y^3 + 30y^2 + 9y + 1)(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)$. The number of irreducible monic polynomial are also 6, 2 of degree 1 and 4 of degree 5.

xxiii. Let $n = 23$, Then $C_i = \{i.31^j mod 23 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$, $C_5 = \{5, 10, 11, 14, 7, 15, 17, 19, 20, 21, 22\}$ There are 3 cyclotomic cosets 31 mod 23. We factorize $y^{23} - 1$. $y^{23} - 1 = (y - 1)(y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)$
$(y^{11} + 8y^{10} + 5y^9 + 27y^8 + 20y^7 + 14y^6 + 18y^5 + 27y^4 + 4y^3 + 10y^2 + 7y + 30)(y^{11} + 24y^{10} + 21y^9 + 27y^8 + 4y^7 + 13y^6 + 17y^5 + 11y^4 + 4y^3 + 26y^2 + 23y + 30)$ The number of irreducible monic polynomial are also 3, 1 of degree 1 and 2 of degree 11. xxiv Let $n = 24$, Then $C_i = \{i.31^j \text{ mod } 24 \in \mathbb{Z}_{31} : j = 0, 1, 2, ..\}$, $C_0 = \{0\}$, $C_1 = \{1, 7\}$, $C_2 = \{2, 14\}$, $C_3 = \{3, 21\}$,$C_4 = \{4\}$, $C_5 = \{5, 11\}$, $C_6 = \{6, 18\}$, $C_8 = \{8\}$, $C_9 = \{9, 15\}$, $C_{12} = \{12\}$, $C_{13} = \{13, 19\}$, $C_{16} = \{16\}$, $C_{10} = \{10, 22\}$, $C_{17} = \{17, 23\}$, $C_{20} = \{20\}$. There are 15 cyclotomic cosets 31 mod 24. We factorize $y^{24} - 1$. $y^{24} - 1 = (y^{12} - 1)(y^{12} + 1) = (y^6 - 1)(y^6 + 1)(y^{12} +) = (y^3 - 1)(y^3 +)(y^6 + 1)(y^{12} + 1) = (y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25)(y^2 + 1)(y^2 + 5)(y^2 + 25)(y^{12} + 1) = (y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25)(y^2 + 1)(y^2 + 5)(y^2 + 25)(y^2 + 8y + 1)(y^2 + 23y + 1)(y^2 +$

$17y+5)(y^2+14y+5)(y^2+22y+25)(y^2+9y+25)$. The number of irreducible monic polynomial are also 15, 6 of degree 1 and 9 of degree 2.

xxv. Let $n=25$, Then $C_i=\{i.31^j \bmod 25 \in \mathbb{Z}_{31} : j=0,1,2,..\}$, $C_0=\{0\}$, $C_1=\{1,6,11,16,21\}$, $C_2=\{2,7,12,17,22\}$, $C_3=\{3,18,8,23,13\}$, $C_4=\{4,24,19,14,9\}$,$C_5=\{5\}$, $C_{10}=\{10\}$, $C_{15}=\{15\}$, $C_{20}=\{20\}$ There are 9 cyclotomic cosets 31 mod 25. We factorize $y^{25}-1$. $y^{25}-1=(y-1)(y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1)=(y+30)(y+29)(y+27)(y+23)(y+15)(y^5+15)(y^5+23)(y^5+29)(y^5+27)$. The number of irreducible monic polynomial are also 9, 5 of degree 1 and 4 of degree 4.

xxvi. Let $n=26$, Then $C_i=\{i.31^j \bmod 26 \in \mathbb{Z}_{31} : j=0,1,2,..\}$, $C_0=\{0\}$, $C_1=\{1,5,21,25\}$, $C_2=\{2,10,16,24\}$, $C_3=\{3,11,15,23\}$, $C_4=\{4,20,22,6\}$ $C_7=\{7,9,19,17\}$, $C_8=\{8,14,18,12\}$, $C_{13}=\{13\}$. There are 8 cyclotomic cosets 31 mod 26. We factorize $y^{26}-1$. $y^{26}-1=(y^{13}-1)(y^{13}+1)=(y-1)(y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1)(y^{13}+1)=(y+30)(y^4+4y^3+11y^2+4y+1)(y^4+6y^3+29y^2+6y+1)(y^4+22y^3+27y^2+22y+1)(y^{13}+1)=(y+1)(y+30)(y^4+4y^3+11y^2+4y+1)(y^4+6y^3+29y^2+6y+1)(y^4+22y^3+27y^2+22y+1)(y^4+9y^3+27y^2+9y+1)(y^4+27y^3+11y^2+27y+1)(y^4+25y^3+29y^2+25y+1)$. The number of irreducible monic polynomial are also 8, 2 of degree 1 and 6 of degree 4.

xxvii. Let $n=27$, Then $C_i=\{i.31^j \bmod 27 \in \mathbb{Z}_{31} : j=0,1,2,..\}$, $C_0=\{0\}$, $C_1=\{1,4,16,10,13,25,19,22,7\}$, $C_2=\{2,8,5,20,26,23,11,17,14\}$, $C_3=\{3,12,21\}$, $C_6=\{6,24,15\}$, $C_9=\{9\}$, $C_{18}=\{18\}$. There are 7 cyclotomic cosets 31 mod 27. We factorize $y^{27}-1$. $y^{27}-1=(y-1)(y^{26}+y^{25}+y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1)=(y+30)(y+6)(y+26)(y^3+6)(y^3+26)(y^9+6)(y^9+16)$. The number of irreducible monic polynomial are also 7, 3 of degree 1, 2 of degree 3 and 2 of degree 9.

xxviii. Let $n=28$, Then $C_i=\{i.31^j \bmod 28 \in \mathbb{Z}_{31} : j=0,1,2,..\}$, $C_0=\{0\}$, $C_1=\{1,3,9,27,25,19\}$, $C_2=\{2,6,18,26,22,10\}$, $C_4=\{4,12,8,24,16,20\}$, $C_5=\{5,15,17,23,13,11\}$, $C_7=\{7,21\}$, $C_{14}=\{14\}$ There are 7 cyclotomic cosets 31 mod 28. We factorize $y^{28}-1$. $y^{28}-1=(y^{14}-1)(y^{14}+1)=(y^7-1)(y^7+1)(y^{14}+1)=(y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y^7+1)(y^{14}+1)=(y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y+1)(y^6+30y^5+y^4+30y^3+y^2+30y+1)(y^{14}+1)=(y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y+1)(y^6+30y^5+y^4+30y^3+y^2+30y+1)(y^2+1)(y^{12}+30y^{10}+y^8+30y^6+y^4+30y^2+1)=(y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y+1)(y^6+30y^5+y^4+30y^3+y^2+30y+1)(y^2+1)(y^6+10y^5+3y^4+10y^3+3y^2+10y+1)(y^6+21y^5+3y^4+21y^3+3y^2+21y+1)$. The number of irreducible monic polynomial are also 7, 2 of degree 1 and 1 of degree 2, 4 of degree 6.

xxix. Let $n=29$, Then $C_i=\{i.31^j \bmod 2 \in \mathbb{Z}_{31} : j=0,1,2,..\}$, $C_0=\{0\}$, $C_1=\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29\}$. There are 2 cyclotomic cosets 31 mod 29. We factorize $y^{29}-1$. $y^{29}-1=(y-1)(y^{28}+y^{27}+y^{26}+y^{25}+y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1)=(y+30)(y^{28}+y^{27}+y^{26}+y^{25}+y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1)$. The number of irreducible monic polynomial are also 2, 1 of degree 1, 1 of degree 28.

xxx Let $n=30$, Then $C_i=\{i.31^j \bmod 15 \in \mathbb{Z}_{31} : j=0,1,2,..\}$, $C_0=\{0\}$, $C_1=\{1\}$, $C_2=\{2\}$, $C_3=\{3\}$, $C_4=\{4\}$,$C_5=\{5\}$, $C_6=\{6\}$, $C_7=\{7\}$, $C_8=\{8\}$, $C_9=\{9\}$, $C_{10}=\{10\}$, $C_{11}=\{11\}$, $C_{12}=\{12\}$, $C_{13}=\{13\}$, $C_{14}=\{14\}$, $C_{15}=\{15\}$, $C_{16}=\{16\}$, $C_{17}=\{17\}$, $C_{18}=\{18\}$, $C_{19}=\{19\}$, $C_{20}=\{20\}$, $C_{21}=\{21\}$, $C_{22}=\{22\}$, $C_{23}=\{23\}$, $C_{24}=\{24\}$, $C_{25}=\{25\}$, $C_{26}=\{26\}$, $C_{27}=\{27\}$, $C_{28}=\{28\}$, $C_{29}=\{29\}$. There are 30 cyclotomic cosets 31 mod 15. We factorize $y^{30}-1$. $y^{30}-1=(y^{15}-1)(y^{15}+1)=(y+30)(y+26)(y+6)(y+3)(y+21)(y+29)(y+23)(y+27)(y+17)(y+11)(y+24)(y+22)(y+12)(y+13)(y+15)(y^{15}+1)=(y+30)(y+26)(y+6)(y+3)(y+21)(y+29)(y+23)(y+27)(y+17)(y+11)(y+24)(y+22)(y+12)(y+13)(y+15)(y+4)(y+5)(y+7)(y+8)(y+9)(y+10)(y+14)(y+16)(y+18)(y+19)(y+20)(y+25)(y+28)(y+1)(y+2)$
The number of irreducible monic polynomial are also 30 of degree 1.

Conjecture 2.1. *Suppose h is the number of cyclotomic coset* 31 *mode n , then the number of cyclic codes over N over $\mathbb{Z}_{31}$ is given by $N=2^h$*

Table 1.The number of cyclic vode of length $n = 1, 2, 3, ..., 30$ over $\mathbb{Z}_{31}$

n	$y^n - 1$	Number of q Cyclotomic Cosets equal h	Number of Cyclic code equal 2^h
1	$y - 1$	1	$2^1 = 2$
2	$y^2 - 1$	2	$2^2 = 4$
3	$y^3 - 1$	3	$2^3 = 8$
4	$y^4 - 1$	3	$2^3 = 8$
5	$y^5 - 1$	5	$2^5 = 32$
6	$y^6 - 1$	6	$2^6 = 64$
7	$y^7 - 1$	2	$2^2 = 4$
8	$y^8 - 1$	5	$2^5 = 32$
9	$y^9 - 1$	5	$2^5 = 32$
10	$y^{10} - 1$	10	$2^{10} = 1,024$
11	$y^{11} - 1$	3	$2^3 = 8$
12	$y^{12} - 1$	9	$2^9 = 512$
13	$y^{13} - 1$	4	$2^4 = 16$
14	$y^{14} - 1$	4	$2^4 = 16$
15	$y^{15} - 1$	15	$2^{15} = 32,768$
16	$y^{16} - 1$	9	$2^9 = 512$
17	$y^{17} - 1$	2	$2^2 = 4$
18	$y^{18} - 1$	10	$2^{10} = 1,024$
19	$y^{19} - 1$	4	$2^4 = 16$
20	$y^{20} - 1$	15	$2^{15} = 32,768$
21	$y^{21} - 1$	6	$2^6 = 64$
22	$y^{22} - 1$	6	$2^6 = 64$
23	$y^{23} - 1$	3	$2^3 = 8$
24	$y^{24} - 1$	15	$2^{15} = 32,768$
25	$y^{25} - 1$	9	$2^9 = 512$
26	$y^{26} - 1$	8	$2^8 = 256$
27	$y^{27} - 1$	7	$2^7 = 128$
28	$y^{28} - 1$	7	$2^7 = 128$
29	$y^{29} - 1$	2	$2^2 = 4$
30	$y^{30} - 1$	30	$2^{30} = 1,073,741,824$

2.2 Number of cyclic codes when $n = 31w$, $1 \leq w \leq 15$

$y^n - 1 = (y - 1)^n = ((y + 30)^n) = (y + 30)^{31w}$

i. Let $w = 1$, then $n = 31$ and $y^{31} - 1 = (y - 1)^{31} = (y + 30)^{31}$. Number of cyclic codes are, $(31 + 1) = 32$.

ii. Let $w = 2$, then $n = 62$ and $y^{62} - 1 = (y^2 - 1)^{31} = (y - 1)^{31}(y + 1)^{31} = (y + 30)^{31}(y + 1)^{31}$. Number of cyclic codes are, $(31 + 1)^2 = 32^2 = 1,024$.

iii. Let $w = 3$, then $n = 93$ and $y^{93} - 1 = (y^3 - 1)^{31} = (y + 30)(y + 26)(y + 6)$

iv. Let $w = 4$, then $n = 124$ and $y^{124} - 1 = (y^4 - 1)^{31} = (y^2 - 1)(y^2 + 1) = (y + 30)^{31}(y + 1)^{31}(y^2 + 1)^{31}$ Number of cyclic codes are, $(31 + 1)^3 = 32^3 = 32,768$.

v. Let $w = 5$, then $n = 155$ and $y^{155} - 1 = (y^5 - 1)^{31} = ((y - 1)(y^4 + y^3 + y^2 + y+))^{31} = ((y + 30)(y + 29)(y + 27)(y + 15)(y + 23))^{31}$ Number of cyclic codes are, $(31 + 1)^5 = 32^5 = 33,554,432$.

vi. Let $w = 6$, then $n = 186$ and $y^{186} - 1 = (y^6 - 1)^{31} = ((y^3 - 1)(y^3 + 1))^{31} = ((y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25))^{31}$ Number of cyclic codes are, $(31 + 1)^6 = 32^6 = 887,503,681$.

vii. Let $w = 7$, then $n = 217$ and $y^{217} - 1 = (y^7 - 1)^{31} = ((y+30)(y^6+y^5+y^4+y^3+y^2+y+1))^{31}$ Number of cyclic codes are, $(31+1)^2 = 32^2 = 1,024$.

viii. Let $w = 8$, then $n = 248$ and $y^{248} - 1 = (y^8 - 1)^{31} = ((y^4 - 1)(y^4 + 1))^{31} = ((y+30)^{31}(y+1)^{31}(y^2+1)^{31}(y^2+8y+1)(y^2+23y+1))^{31}$ Number of cyclic codes are, $(31+1)^5 = 32^5 = 28,629,151$.

ix. Let $w = 9$, then $n = 279$ and $y^{279} - 1 = (y^9 - 1)^{31} = ((y+30)(y+26)(y+6)(y^3+6)(y^3+26))^{31}$ Number of cyclic codes are, $(31+1)^5 = 32^5 = 28,629,151$.

x. Let $w = 10$, then $n = 310$ and $y^{310} - 1 = (y^{10} - 1)^{31} = ((y^5 - 1)(y^5 + 1))^{31} = ((y+30)(y+29)(y+27)(y+15)(y+23)(y+1)(y+2)(y+4)(y+8)(y+16))^{31}$ Number of cyclic codes are, $(31+1)^{10} = 32^{10} = 819,628,286,980,801$.

xi. Let $w = 11$, then $n = 341$ and $y^{341} - 1 = (y^{11} - 1)^{31} = ((y-1)(y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1))^{31} = ((y+30)(y^5+10y^4+30y^3+y^2+9y+30)(y^5+22y^4+30y^3+y^2+21y+30))^{31}$ Number of cyclic codes are, $(31+1)^3 = 32^3 = 29,791$.

xii. Let $w = 12$, then $n = 372$ and $y^{372} = (y^{12} - 1)^{31} = ((y^6 - 1)(y^6 + 1))^{31} = ((y+30)(y+26)(y+6)(y+1)(y+5)(y+25)(y^2+1)(y^2+5)(y^2+25))^{31}$ Number of cyclic codes are, $(31+1)^9 = 32^9 = 26,439,622,160,671$.

xiii. Let $w = 13$, then $n = 403$ and $y^{403} - 1 = (y^{13} - 1)^{31} = ((y+30)(y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1))^{31}$
Number of cyclic codes are, $(31+1)^4 = 32^4 = 923,521$.

xiv. Let $w = 14$, then $n = 434$ and $y^{434} - 1 = (y^{14} - 1)^{31} = ((y^7 - 1)(y^7 - 1))^{31} = ((y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y^7+1))^{31} = ((y+30)(y^6+y^5+y^4+y^3+y^2+y+1)(y+1)$
$(y^6+30y^5+y^4+30y^3+y^2+30y+1))^{31}$ Number of cyclic codes are, $(31+1)^4 = 32^4 = 923,521$.

xv. Let $w = 15$, then $n = 465$ and $y^{465} - 1 = (y^{15} - 1)^{31} = ((y-1)(y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1))^{31} = ((y+30)(y+26)(y+6)(y+3)(y+21)(y+29)(y+23)(y+27)(y+17)(y+11)(y+24)(y+22)(y+12)(y+13)(y+15))^{31}$ Number of cyclic codes are, $(31+1)^{15} = 32^{15} = 23,465,261,991,844,685,929,951$.

The Table 2 gives summary of the number of cyclic code

Table 2. The number of cyclic code for $n = 31w$ over $\mathbb{Z}_{31}$

w	$n = 31w$	Number of Factors	Number of Cyclic code
1	31	1	$32^1 = 32$
2	62	2	$32^2 = 1,024$
3	93	3	$32^3 = 32,768$
4	124	3	$32^3 = 32,768$
5	155	5	$32^5 = 33,554,432$
6	186	6	$32^6 = 1,073,741,824$
7	217	2	$32^2 = 1,0244$
8	248	5	$32^5 = 33,554,432$
9	279	5	$32^5 = 33,554,432$
10	310	10	$32^{10} = 1,125,899,906,842,624$
11	341	3	$32^3 = 32,768$
12	372	9	$32^9 = 35,184,372,088,832$
13	403	4	$32^4 = 1,048,576$
14	434	4	$32^4 = 1,048,576$
15	465	15	$32^{15} = 32,768$

Conjecture 2.2. *Suppose $n = 31w, w \in \mathbb{Z}^+$ and $y^n - 1$ factorizes into f linear factors then the number of cyclic code N over $\mathbb{Z}_{31}$ is given by; $N = 32^f$*

2.3 Number of Cyclic Codes when $n = 31^w$, $1 \leq w \leq 10$

i. Let $w = 1$, then $n = 31$ and $y^{31} - 1 = (y-1)^{31} = (y+30)^{31}$. Number of cyclic codes are, $31^1 + 1 = 31 + 1 = 32$

ii. Let $w = 2$, then $n = 31^2$ and $y^{31^2} - 1 = (y-1)^{31^2} = (y+30)^{31^2}$ Number of cyclic codes are, $31^2 + 1 = 961 + 1 = 962$

iii. Let $w = 3$, then $n = 31^3$ and $y^{31^3} - 1 = (y-1)^{31^3} = (y+30)^{31^3}$ Number of cyclic codes are, $31^3 + 1 = 29,791 + 1 = 29,792$

iv. Let $w = 4$, then $n = 31^4$ and $y^{31^4} - 1 = (y-1)^{31^4} = (y+30)^{31^4}$. Number of cyclic codes are, $31^4 + 1 = 923,521 + 1 = 923,522$

v. Let $w = 5$, then $n = 31^5$ and $y^{31^5} - 1 = (y-1)^{31^5} = (y+30)^{31^5}$ Number of cyclic codes are, $31^5 + 1 = 31 + 1 = 28,629,152$

vi. Let $w = 6$, then $n = 31^6$ and $y^{31^6} - 1 = (y-1)^{31^6} = (y+30)^{31^6}$ Number of cyclic codes are, $31^6 + 1 = 887,503,681 + 1 = 887,503,682$

vii. Let $w = 7$, then $n = 31^7$ and $y^{31^7} - 1 = (y-1)^{31^7} = (y+30)^{31^7}$ Number of cyclic codes are, $31^7 + 1 = 27,512,614,111 + 1 = 27,512,614,112$

viii. Let $w = 8$, then $n = 31^8$ and $y^{31^8} - 1 = (y-1)^{31^8} = (y+30)^{31^8}$ Number of cyclic codes are, $31^8 + 1 = 852,891,037,441 + 1 = 852,891,037,442$

ix. Let $w = 9$, then $n = 31^9$ and $y^{31^9} - 1 = (y-1)^{31^9} = (y+30)^{31^9}$ Number of cyclic codes are, $31^9 + 1 = 26,439,622,160,671 + 1 = 26,439,622,160,672$

x. Let $w = 10$, then $n = 31^{10}$ and $y^{31^{10}} - 1 = (y-1)^{31^{10}} = (y+30)^{31^{10}}$ Number of cyclic codes are, $31^{10} + 1 = 819,628,286,980,801 + 1 = 819,628,286,980,802$

The Table 3 gives summary of the number of cyclic code.

Table 3. The number of cyclic code for $n = 31^w$ over $\mathbb{Z}_{31}$

w	$n = 31^w$	Factors	Number of Cyclic code
1	31^1	$(y-1)^{31^1}$	$31^1 + 1$
2	31^2	$(y-1)^{31^2}$	$31^2 + 1$
3	31^3	$(y-1)^{31^3}$	$31^3 + 1$
4	31^4	$(y-1)^{31^4}$	$31^4 + 1$
5	31^5	$(y-1)^{31^5}$	$31^5 + 1$
6	31^6	$(y-1)^{31^6}$	$31^6 + 1$
7	31^7	$(y-1)^{31^7}$	$31^7 + 1$
8	31^8	$(y-1)^{31^8}$	$31^8 + 1$
9	31^9	$(y-1)^{31^9}$	$31^9 + 1$
10	31^{10}	$(y-1)^{31^{10}}$	$31^{10} + 1$

In general one can easily infer that for any positive w then, $y^{31^w}-1=(y-1)^{31^w}=(y+30)^{31^w}$. and the number of cyclic codes is given by 31^w+1.

Conjecture 2.3. *Suppose $n=31^w, w\in\mathbb{Z}^+$ and y^n-1 factorizes into f linear factors such that $y^n-1=(y-1)^n=(y-1)^{31^w}$ then the number of cyclic code N over $\mathbb{Z}_{31}$ is given by $N=31^w+1$*

2.4 Number of cyclic codes when $n=q.31^r$ where $r=0,1,2,3,..,$and q is prime

i. Let $q=2$ then $n=2(31)^r$ for $r=0,1,2,....$ and $y^{2(31)^r}-1=(y^2-1)^{31^r}=(y+30)^{31^r}(y+1)^{31^r}$
when $r=0$, $(y^2-1)^{31^0}=y^2-1=(y+30)(y+1)$. The number of factors f equals $(31^0+1)=2$ and the number cyclic codes $=f^2=4$
When $r=1$, $(y^2-1)^{31^1}=(y^2-1)^{31}=((y+30)(y+1))^{31}=(y+30)^{31}(y+1)^{31}$, number of factors f equals $31^1+1=32$ number of cyclic codes $=(31+1)^2=1024$
when $r=2$, $y^n-1=(y+30)^{31^2}(y+1)^{31^2}$, number of factors f equals $31^2+1=62$ number of cyclic codes $=(31^2+1)^2=925,444$
In general, $y^{2(31^r)}-1=[(y-1)^{31^r}]^2$. number of factors f equals $=31^r+1$ and number of cyclic codes $=(31^r+1)^2$

ii. Let $q=3$ then $n=3(31)^r$ for $r=0,1,2,....$ and $y^n-1=y^{3(31^r)}-1=(y^3-1)^{31^r}=(y+30)^{31^r}(y+26)^{31^r}(y+6)^{31^r}$
When $r=0$, $y^n-1=(y+30)^{31^0}(y+26)^{31^0}(y+6)^{31^0}$. Number of cyclic codes $=(31^0+1)^3=8$.
When $r=1$, $y^n-1=(y+30)^{31^1}(y+26)^{31^1}(y+6)^{31^1}$ Number of cyclic codes $=(31^1+1)^3=32,768$.
When $r=2$, $y^n-1=(y+30)^{31^2}(y+26)^{31^2}(y+6)^{31^2}$. Number of cyclic codes $=(31^2+1)^3$. In general, number of cyclic codes $=(31^r+1)^3$.

iii. Let $q=5$ then $n=5(31)^r, r=0,1,2,....$ and $y^n-1=y^{5(31)^r}-1=(y^5-1)^{31^r}=(y+30)^{31^r}(y+29)^{31^r}(y+27)^{31^r}(y+15)^{31^r}(y+23)^{31^r}$
When $r=0$, $y^n-1=(y+30)(y+29)(y+27)(y+15)(y+23)$. Number of cyclic codes $=(31^0+1)^5=32$.
When $r=1$, $y^n-1=(y+30)^{31}(y+29)^{31}(y+27^{31})(y+15)^{31}(y+23)^{31}$. Number of cyclic codes $=(31^1+1)^5$
When $r=2$, $y^n-1=(y+30)^{31^2}(y+29)^{31^2}(y+27^{31^2})(y+15)^{31^2}(y+23)^{31^2}$. Number of cyclic codes $=(31^2+1)^5$.
In general, number of cyclic codes $=(31^r+1)^5$.

iv. Let $q=7$ then $n=7(31)^r, r=0,1,2,....$ and $y^n-1=y^{7(31)^r}-1=(y^7-1)^{31^r}$
$=(y+30)^{31^r}(y^6+y^5+y^4+y^3+y^4+y^3+y^2+y+1)^{31^r}$.
When $r=0$, $y^n-1=(y+30)^{31^0}(y^6+y^5+y^4+y^3+y^4+y^3+y^2+y+1)^{31^0}$. Number of cyclic codes $=(31^0+1)^2$
When $r=1$, $y^n-1=(y+30)^{31^1}(y^6+y^5+y^4+y^3+y^4+y^3+y^2+y+1)^{31^1}$ Number of cyclic codes $=(31^1+1)^2$.
When $r=2$, $y^n-1=(y+30)^{31^2}(y^6+y^5+y^4+y^3+y^4+y^3+y^2+y+1)^{31^2}$ Number of cyclic codes $=(31^2+1)^2$. In general, number of cyclic codes $=(31^r+1)^2$.

v. Let $q=11$ then $n=11(31)^r, r=0,1,2,....$ and $y^n-1=y^{11(31)^r}-1=(y^{11}-1)^{31^r}=(y+30)^{31^r}(y^5+10y^4+30y^3+y^2+9y+30)^{31^r}(y^5+22y^4+30y^3+y^2+21y+30)^{31^r}$.
When $r=0$, $y^n-1=(y+30)^{31^0}(y^5+10y^4+30y^3+y^2+9y+30)^{31^0}(y^5+22y^4+30y^3+y^2+21y+30)^{31^0}$
Number of cyclic codes $=(31^0+1)^3$.
When $r=1$, $y^n-1=(y+30)^{31^1}(y^5+10y^4+30y^3+y^2+9y+30)^{31^1}(y^5+22y^4+30y^3+y^2+21y+30)^{31^1}$
Number of cyclic codes $=(31^1+1)^3$.
When $r=2$, $y^n-1=(y+30)^{31^2}(y^5+10y^4+30y^3+y^2+9y+30)^{31^2}(y^5+22y^4+30y^3+y^2+21y+30)^{31^2}$
Number of cyclic codes $=(31^2+1)^3$.

vi. Let $q=13$ then $n=13(31)^r, r=0,1,2,....$ and $y^n-1=y^{13(31)^r}-1=(y^{13}-1)^{31^r}=(y+30)^{31^r}(y^4+4y^3+11y^2+4y+)^{31^r}(y^4+6y^3+29y^2+6y+1)^{31^r}(y^4+22y^3+28y^2+22y+1)^{31^r}$

When $r = 0$, $y^n - 1 = (y+30)^{31^0}(y^4+4y^3+11y^2+4y+)^{31^0}(y^4+6y^3+29y^2+6y+1)^{31^0}$ $(y^4+22y^3+28y^2+22y+1)^{31^0}$. Number of cyclic codes $= (31^0+1)^4$.
When $r = 1$, $y^n - 1 = (y+30)^{31^1}(y^4+4y^3+11y^2+4y+)^{31^1}(y^4+6y^3+29y^2+6y+1)^{31^1}$ $(y^4+22y^3+28y^2+22y+1)^{31^1}$. Number of cyclic codes $= (31^1+1)^4$.
When $r = 2$, $y^n - 1 = (y+30)^{31^2}(y^4+4y^3+11y^2+4y+)^{31^2}(y^4+6y^3+29y^2+6y+1)^{31^2}$ $(y^4+22y^3+28y^2+22y+1)^{31^2}$. Number of cyclic codes $= (31^2+1)^4$

vii. Let $q = 17$ then $n = 17(31)^r, r = 0,1,2,....$ and $y^n - 1 = y^{17(31)^r} - 1 = (y^{17}-1)^{31^r}$ $= (y+30)^{31^r}(y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6$ $+y^5+y^4+y^3+y^2+y+1)^{31^r}$.
When $r = 0$, $y^n - 1 = (y+30)^{31^0}(y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6$ $+y^5+y^4+y^3+y^2+y+1)^{31^0}$. Number of cyclic codes $= (31^0+1)^2$.
When $r = 1$, $y^n - 1 = (y+30)^{31^1}(y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6$ $+y^5+y^4+y^3+y^2+y+1)^{31^1}$. Number of cyclic codes $= (31^1+1)^2$. When $r = 2$, $y^n - 1 = (y+30)^{31^2}(y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6$ $+y^5+y^4+y^3+y^2+y+1)^{31^2}$. Number of cyclic codes $= (31^2+1)^2$.

viii. Let $q = 19$ then $n = 19(31)^r, r = 0,1,2,....$ and $y^n - 1 = y^{19(31)^r} - 1 = (y^{19}-1)^{31^r}$ $= (y+30)^{31^r}(y^6+4y^5+14y^4+18y^3+14y^2+4y+1)^{31^r}$ $(y^6+5y^5+25y^4+y^3+5y^2+25y+1)^{31^r}(y^6+25y^5+5y^4+y^3+25y^3+5y+1)^{31^r}$.
when $r = 0$, $y^n - 1 = (y+30)^{31^0}(y^6+4y^5+14y^4+18y^3+14y^2+4y+1)^{31^0}$ $(y^6+5y^5+25y^4+y^3+5y^2+25y+1)^{31^0}(y^6+25y^5+5y^4+y^3+25y^3+5y+1)^{31^0}$. Number of cyclic codes $= (31^0+1)^4$.
When $r = 1$, $y^n - 1 = (y+30)^{31^1}(y^6+4y^5+14y^4+18y^3+14y^2+4y+1)^{31^1}$ $(y^6+5y^5+25y^4+y^3+5y^2+25y+1)^{31^1}(y^6+25y^5+5y^4+y^3+25y^3+5y+1)^{31^1}$.
Number of cyclic codes $= (31^0+1)^4$.
When $r = 2$, $y^n - 1 = (y+30)^{31^2}(y^6+4y^5+14y^4+18y^3+14y^2+4y+1)^{31^2}$ $(y^6+5y^5+25y^4+y^3+5y^2+25y+1)^{31^2}(y^6+25y^5+5y^4+y^3+25y^3+5y+1)^{31^2}$. Number of cyclic codes $= (31^2+1)^4$.

ix. Let $q = 23$ then $n = 23(31)^r, r = 0,1,2,....$ and $y^n - 1 = y^{23(31)^r} - 1 = (y^{23}-1)^{31^r} = (y+30)^{31^r}(y^{11}+8y^{10}+5y^9+27y^8+20y^7+14y^6+18y^5+27y^4+4y^3+10y^2+7y+30)^{31^r}(y^{11}+24y^{10}+21y^9+27y^8+4y^7+13y^6+17y^5+11y^4+4y^3+26y^2+23y+30)^{31^r}$.
When $r = 0$, $y^n - 1 = (y+30)^{31^0}(y^{11}+8y^{10}+5y^9+27y^8+20y^7+14y^6+18y^5+27y^4+4y^3+10y^2+7y+30)^{31^0}(y^{11}+24y^{10}+21y^9+27y^8+4y^7+13y^6+17y^5+11y^4+4y^3+26y^2+23y+30)^{31^0}$. Number of cyclic codes $= (31^0+1)^3$.
When $r = 1$, $y^n - 1 = (y+30)^{31^1}(y^{11}+8y^{10}+5y^9+27y^8+20y^7+14y^6+18y^5+27y^4+4y^3+10y^2+7y+30)^{31^1}(y^{11}+24y^{10}+21y^9+27y^8+4y^7+13y^6+17y^5+11y^4+4y^3+26y^2+23y+30)^{31^1}$. Number of cyclic codes $= (31^1+1)^3$.
When $r = 2$, $y^n - 1 = (y+30)^{31^2}(y^{11}+8y^{10}+5y^9+27y^8+20y^7+14y^6+18y^5+27y^4+4y^3+10y^2+7y+30)^{31^2}(y^{11}+24y^{10}+21y^9+27y^8+4y^7+13y^6+17y^5+11y^4+4y^3+26y^2+23y+30)^{31^2}$. Number of cyclic codes $= (31^2+1)^3$.

x. Let $q = 29$ then $n = 29(31)^r, r = 0,1,2,....$ and $y^n - 1 = y^{29(31)^r} - 1 = (y^{29}-1)^{31^r}(y^{29}-1)^{31^r} = ((y+30)(y^{28}+y^{27}+y^{26}+y^{25}+y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1))^{31^r}$.
When $r = 0$, $(y^{29}-1) = ((y+30)(y^{28}+y^{27}+y^{26}+y^{25}+y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1))$. Number of cyclic codes $(31^0+1)^2$.
When $r = 1$, $(y^{29}-1)^{31} = ((y+30)(y^{28}+y^{27}+y^{26}+y^{25}+y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1))^{31}$. Number of cyclic

codes $(31^1+1)^2$ When $r=2$, $(y^{29}-1)^{31^2} = ((y+30)(y^{28}+y^{27}+y^{26}+y^{25}+y^{24}+y^{23}+y^{22}+y^{21}+y^{20}+y^{19}+y^{18}+y^{17}+y^{16}+y^{15}+y^{14}+y^{13}+y^{12}+y^{11}+y^{10}+y^{9}+y^{8}+y^{7}+y^{6}+y^{5}+y^{4}+y^{3}+y^{2}+y+1))^{31^2}$
Number of cyclic codes $(31^2+1)^2$

The Table 4 gives summary of the number of cyclic code.

Table 4. The number of cyclic code for $n = q.31^r$ over $\mathbb{Z}_{31}$

q	$n = q.31^r$	Factors	number of factors in $y^q - 1$	Number of Cyclic code
2	2.31^r	$(y^2-1)^{31^r}$	2	$(31^r+1)^2$
3	3.31^r	$(y^3-1)^{31^r}$	3	$(31^r+1)^3$
5	5.31^r	$(y^5-1)^{31^r}$	5	$(31^r+1)^5$
7	7.31^r	$(y^7-1)^{31^r}$	2	$(31^r+1)^2$
11	11.31^r	$(y^{11}-1)^{31^r}$	3	$(31^r+1)^3$
13	13.31^r	$(y^{13}-1)^{31^r}$	4	$(31^r+1)^4$
17	17.31^r	$(y^{17}-1)^{31^r}$	2	$(31^r+1)^2$
19	19.31^r	$(y^{19}-1)^{31^r}$	4	$(31^r+1)^4$
23	23.31^r	$(y^{23}-1)^{31^r}$	3	$(31^r+1)^3$
29	29.31^r	$(y^{29}-1)^{31^r}$	2	$(31^r+1)^2$

Conjecture 2.4. *Suppose $n = q.31^r, r = 0,1,2,..$, q prime and $y^n - 1 = (y-1)^n = (y-1)^{q.31^r}$ factorizes into 31^r+1 linear factors then the number of cyclic code N over $\mathbb{Z}_{31}$ is given by $N = (31^r+1)^q$*

2.5 Number of cyclic codes when $n = q^r.31^r$ where $r = 1,2,3,..,$and q is prime

i. Let $q=2$ then, $n = 2^r(31)^r$ for $r = 1,2,....$ and $y^{2^r(31)^r} - 1 = (y^{2^r}-1)^{31^r}$.
When $r=1$, $(y^{2^1}-1)^{31^1} = (y^2-1)^{31} = ((y+30)(y+1))^{31} = (y+30)^{31}(y+1)^{31}$,
then the number of cyclic codes $= (31+1)^2 = 1024$.
When $r=2$, $y^n - 1 = (y^{2^2}-1)^{31^2}$, then the number of cyclic codes $= (31^2+1)^{2^2}$.
In general, number of cyclic codes $= (31^r+1)^{2^r}$.

ii. Let $q=3$ then $n = 3^r(31)^r$ for $r = 1,2,....$ and $y^n - 1 = y^{3^r(31^r)} - 1 = (y^{3^r}-1)^{31^r}$.
When $r=1$, $y^n - 1 = y^{3^1(31^1)} - 1 = (y^{3^1}-1)^{31^1}$,
then the number of cyclic codes $= (31^1+1)^{3^1} = 32,768$.
When $r=2$, $y^n - 1 = (y^{3^2}-1)^{31^2}$, then the number of cyclic codes $= (31^2+1)^{3^2}$.
In general, number of cyclic codes $= (31^r+1)^{3^r}$.

iii. Let $q=5$ then $n = 5^r(31)^r, r = 1,2,....$ and $y^n - 1 = y^{5^r(31)^r} - 1 = (y^{5^r}-1)^{31^r}$.
When $r=1$, $y^n - 1 = (y^{5^1}-1)^{31^1}$, then the number of cyclic codes $= (31^1+1)^{5^1}$.
When $r=2$, $y^n - 1 = (y^{5^2}-1)^{31^2}$, then the number of cyclic codes $= (31^2+1)^{5^2}$.
In general, number of cyclic codes $= (31^r+1)^{5^r}$.

iv. Let $q=7$ then $n = 7^r(31)^r, r = 1,2,....$ and $y^n - 1 = y^{7^r(31)^r} - 1 = (y^{7^r}-1)^{31^r}$.
When $r=1$, $y^n - 1 = (y^{7^1}-1)^{31^1}$, then the number of cyclic codes $= (31^1+1)^{2^1}$.
When $r=2$, $y^n - 1 = (y^{7^2}-1)^{31^2}$, then the number of cyclic codes $= (31^2+1)^{2^2}$
In general, number of cyclic codes $= (31^r+1)^{2^r}$.

v. Let $q=11$ then $n = 11^r(31)^r, r = 1,2,....$ and $y^n - 1 = y^{11^r(31)^r} - 1 = (y^{11^r}-1)^{31^r}$.
When $r=1$, $y^n - 1 = (y^{11^1}-1)^{31^1}$, then the number of cyclic codes $= (31^1+1)^{3^1}$.
When $r=2$, $y^n - 1 = (y^{11^2}-1)^{31^2}$, then the number of cyclic codes $= (31^2+1)^{3^2}$.
In general, number of cyclic codes $= (31^r+1)^{3^r}$.

vi. Let $q = 13$ then $n = 13^r(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{13^r(31)^r} - 1 = (y^{13^r} - 1)^{31^r}$.
When $r = 1$, $y^n - 1 = (y^{13^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{4^1}$.
When $r = 2$, $y^n - 1 = (y^{13^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{4^2}$.
In general, number of cyclic codes $= (31^r + 1)^{4^r}$.

vii. Let $q = 17$ then $n = 17^r(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{17^r(31)^r} - 1 = (y^{17^r} - 1)^{31^r}$.
When $r = 1$, $y^n - 1 = (y^{17^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{2^1}$
When $r = 2$, $y^n - 1 = (y^{17^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{2^2}$.
In general, number of cyclic codes $= (31^r + 1)^{2^r}$.

viii. Let $q = 19$ then $n = 19^r(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{19^r(31)^r} - 1 = (y^{19^r} - 1)^{31^r}$.
When $r = 1$, $y^n - 1 = (y^{19^1} - 1)^{31^1}$. then the number of cyclic codes $= (31^1 + 1)^{4^1}$
When $r = 2$, $y^n - 1 = (y^{19^2} - 1)^{31^2}$. then the number of cyclic codes $= (31^2 + 1)^{4^2}$.
In general, number of cyclic codes $= (31^r + 1)^{4^r}$.

ix. Let $q = 23$ then $n = 23^r(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{23^r(31)^r} - 1 = (y^{23^r} - 1)^{31^r}$.
When $r = 1$, $y^n - 1 = (y^{23^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{3^1}$.
When $r = 2$, $y^n - 1 = (y^{23^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{3^2}$.
In general, number of cyclic codes $= (31^r + 1)^{3^r}$.

x. Let $q = 29$ then $n = 29^r(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{29^r(31)^r} - 1 = (y^{29^r} - 1)^{31^r}$
When $r = 1$, $y^n - 1 = (y^{29^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{2^1}$.
When $r = 2$, $y^n - 1 = (y^{29^2} - 1)^{31^2}$. then the number of cyclic codes $= (31^2 + 1)^{2^2}$,
In general, number of cyclic codes $= (31^r + 1)^{2^r}$.

The Table 5 gives summary of the number of cyclic code.

Table 5. The number of cyclic code for $n = q^r.31^r$ over $\mathbb{Z}_{31}$

q	$n = q^r.31^r$	Factors	number of factors in $y^q - 1$	Number of Cyclic code
2	$2^r.31^r$	$(y^{2^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$
3	$3^r.31^r$	$(y^{3^r} - 1)^{31^r}$	3	$(31^r + 1)^{3^r}$
5	$5^r.31^r$	$(y^{5^r} - 1)^{31^r}$	5	$(31^r + 1)^{5^r}$
7	$7^r.31^r$	$(y^{7^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$
11	$11^r.31^r$	$(y^{11^r} - 1)^{31^r}$	3	$(31^r + 1)^{3^r}$
13	$13^r.31^r$	$(y^{13^r} - 1)^{31^r}$	4	$(31^r + 1)^{4^r}$
17	$17^r.31^r$	$(y^{17^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$
19	$19^r.31^r$	$(y^{19^r} - 1)^{31^r}$	4	$(31^r + 1)^{4^r}$
23	$23^r.31^r$	$(y^{23^r} - 1)^{31^r}$	3	$(31^r + 1)^{3^r}$
29	$29^r.31^r$	$(y^{29^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$

Conjecture 2.5. *Suppose $n = q^r.31^r, r = 1, 2, .., q$ prime and $y^n - 1 = (y - 1)^n = (y - 1)^{q^r.31^r}$ is such that the number of factors in $y^q - 1$ is f then the number of cyclic code N over $\mathbb{Z}_{31}$ is given by $N = (31^r + 1)^{f^r}$*

2.6 Number of cyclic codes when $n = q^k.31^r$ where $r = 1, 2, 3, ..$,and $k = 1, 2, ...$ q is prime

i. Let $q = 2$ the $n = 2^k(31)^r$ for $r = 1, 2, ...$ and $y^{2^k(31)^r} - 1 = (y^{2^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{2^1(31)^r} - 1 = y^{2(31)^r} - 1$. When $r = 1$, $(y^2 - 1)^{31^1} = (y^2 - 1)^{31} = ((y + 30)(y + 1))^{31} = (y + 30)^{31}(y + 1)^{31}$. Number of cyclic codes $= (31 + 1)^2 = 1024$. When $r = 2$,

$y^n - 1 = (y^2 - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$. In general, $y^{2^r(31^r)} - 1 = [(y-1)^{31^r}]^2$ and Number of cyclic codes $= (31^r + 1)^2$

ii. Let $q = 3$ then $n = 3^k(31)^r$ for $r = 1, 2, ...$ and $y^n - 1 = y^{3^r(31^r)} - 1 = (y^{3^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{3^1(31)^r} - 1 = y^{3(31)^r} - 1$. When $r = 1$, $y^n - 1 = y^{3(31^1)} - 1 = (y^3 - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^3 = 32,768$. When $r = 2$, $y^n - 1 = (y^3 - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^3$. In general, number of cyclic codes $= (31^r + 1)^3$

iii. Let $q = 5$ then $n = 5^r(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{5^k(31)^r} - 1 = (y^{5^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{5^1(31)^r} - 1 = y^{5(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^5 - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^5$. When $r = 2$, $y^n - 1 = (y^5 - 1)^{31^2}$. number of cyclic codes $= (31^2 + 1)^5$. In general, number of cyclic codes $= (31^r + 1)^5$.

iv. Let $q = 7$ then $n = 7^k(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{7^k(31)^r} - 1 = (y^{7^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{7^1(31)^r} - 1 = y^{7(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^7 - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^2$. When $r = 2$, $y^n - 1 = (y^{7^2} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$. In general, number of cyclic codes $= (31^r + 1)^2$.

v. Let $q = 11$ then $n = 11^k(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{11^k(31)^r} - 1 = (y^{11^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{11^1(31)^r} - 1 = y^{11(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^{11} - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^3$. When $r = 2$, $y^n - 1 = (y^{11} - 1)^{31^2}$. Number of cyclic codes $= (31 + 1)^{3^2}$. In general, number of cyclic codes $= (31^r + 1)^{3^r}$.

vi. Let $q = 13$ then $n = 13^k(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{13^k(31)^r} - 1 = (y^{13^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{13^1(31)^r} - 1 = y^{13(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^{13} - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^4$. When $r = 2$, $y^n - 1 = (y^{13} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^4$. In general, number of cyclic codes $= (31^r + 1)^4$

vii. Let $q = 17$ then $n = 17^k(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{17^k(31)^r} - 1 = (y^{17^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{17^1(31)^r} - 1 = y^{17(31)^r} - 1..$ When $r = 1$, $y^n - 1 = (y^{17} - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^2$. When $r = 2$, $y^n - 1 = (y^{17} - 1)^{31^2}$ Number of cyclic codes $= (31^2 + 1)^2$. In general, number of cyclic codes $= (31^r + 1)^2$

viii. Let $q = 19$ then $n = 19^k(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{19^k(31)^r} - 1 = (y^{19^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{19^1(31)^r} - 1 = y^{19(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^{19} - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^4$. When $r = 2$, $y^n - 1 = (y^{19} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^4$. In general, number of cyclic codes $= (31^r + 1)^4$

ix. Let $q = 23$ then $n = 23^k(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{23^k(31)^r} - 1 = (y^{23^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{23^1(31)^r} - 1 = y^{23(31)^r} - 1$
When $r = 1$. $y^n - 1 = (y^{23} - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^3$. When $r = 2$, $y^n - 1 = (y^{23} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^3$. In general, number of cyclic codes $= (31^r + 1)^3$.

x. Let $q = 29$ then $n = 29^r(31)^r, r = 1, 2, ...$ and $y^n - 1 = y^{29^k(31)^r} - 1 = (y^{29^k} - 1)^{31^r}$
Let $k = 1$ be fixed then $y^n - 1 = y^{29^1(31)^r} - 1 = y^{29(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^{29} - 1)^{31}$. Number of cyclic codes $= (31 + 1)^2$. When $r = 2$, $y^n - 1 = (y^{29} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$. In general, number of cyclic codes $= (31^r + 1)^2$.

The Table 6 gives summary of the number of cyclic code.

Conjecture 2.6. *Suppose* $n = q^k.31^r, r, k \in \mathbb{Z}^+$ *, q prime and* $y^n - 1 = (y-1)^n = (y-1)^{q^k.31^r}$ *is such that the number of factors in* $y^q - 1$ *is f then for a fixed k the number of cyclic code N over* $\mathbb{Z}_{31}$ *is given by* $N = (31^r + 1)^{f^k}$

Table 6. The number of cyclic code for $n = q^k.31^r$ over $\mathbb{Z}_{31}$

q	$n = q^k.31^r$	Factors	number of factors in $y^q - 1$	Number of Cyclic code
2	$2^k.31^r$	$(y^{2^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$
3	$3^k.31^r$	$(y^{3^k} - 1)^{31^r}$	3	$(31^r + 1)^{3^k}$
5	$5^k.31^r$	$(y^{5^k} - 1)^{31^r}$	5	$(31^r + 1)^{5^k}$
7	$7^k.31^r$	$(y^{7^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$
11	$11^k.31^r$	$(y^{11^k} - 1)^{31^r}$	3	$(31^r + 1)^{3^k}$
13	$13^k.31^r$	$(y^{13^k} - 1)^{31^r}$	4	$(31^r + 1)^{4^k}$
17	$17^k.31^r$	$(y^{17^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$
19	$19^k.31^r$	$(y^{19^k} - 1)^{31^r}$	4	$(31^r + 1)^{4^k}$
23	$23^k.31^r$	$(y^{23^k} - 1)^{31^r}$	3	$(31^r + 1)^{3^k}$
29	$29^k.31^r$	$(y^{29^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$

3 Conclusion

In this study, we partitioned cyclotomic polynomial $y^n - 1$ using cyclotomic cosets 31 mod n and factorize $y^n - 1$ using case to case basis. We then investigated the number of cyclic codes for each case. The following conclusions were made.

1. The number of cyclotomic cosets equals to the number of irreducible monic factors in $y^n - 1$
2. Let h be the number of cyclotomic cosets, then the number of cyclic codes equal to 2^h
3. Let $\mathbb{Z}_{31}$ be a given finite field, $y^n - 1$ be cyclotomic polynomial and f be the number of irreducible factors in $y^n - 1$. Then the number of cyclic codes N over $\mathbb{Z}_{31}$ is given by;

$$N = \begin{cases} (31+1)^f & : \quad n = 31w, w \in \mathbb{Z}^+ \\ 31^w + 1 & : \quad n = 31^w, w \in \mathbb{Z}^+ \\ (31^r + 1)^f & : \quad n = q.31^r, r = 0, 1, 2..., qisprime \\ (31^r + 1)^{f^r} & : \quad n = q^r.31^r, r \in \mathbb{Z}^+ \\ (31^r + 1)^{f^k} & : \quad n = q^k.31^r, k, r \in \mathbb{Z}^+ \end{cases}$$

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

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ISSN 2456-9968.

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