International Astronomy and Astrophysics Research Journal

Volume 4, Issue 4, Page 49-64, 2022; Article no.IAARJ.94796

Out-of-plane Equilibrium Points in the ER3BP with Triaxial-radiating Primaries with a P-R Drag Force Surrounded by a Belt

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/94796

Original Research Article

Received: 25/10/2022
Accepted: 29/12/2022
Published: 31/12/2022

ABSTRACT

This paper studies the motion of an infinitesimal particle near the out-of-plane equilibrium points in the elliptic restricted three body problem (ER3BP) when the primaries are triaxial rigid bodies, sources of radiation with a Poynting-Robertson (P-R) drag force surrounded by a belt. It is observed that there exist two out-of-plane equilibria which lie in the ξζ- plane in symmetrical positions with respect to the orbital plane. The parameters involved in the system affect their positions. The position changes with an increase in triaxiality, radiation and belt in the presence of P-R drag force. We found that for the binary system the effect of triaxialty and the belt moves the out-of-plane equilibrium points in opposite directions. The position and linear stability of the out-of-plane equilibrium points are investigated numerically using first, arbitrary values for the parameters and then for the two binary systems (Xi-Bootis and Kruger 60) and they are found to be unstable in each case.

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Keywords: Triaxiality; radiation; elliptic restricted three body problem; stability; gravitational poenial from the belt; binary systems.

1. INTRODUCTION

Till today, the three-body problem remains one of the most important problem in celestial mechanics. It has been one of the most researched field in space dynamics and astrophysics. Well known mathematicians and scientists have obtained interesting and significant results when studying and predicting the motion of natural bodies.

Leonhard Euler in 1765 modified the three body problem into the restricted three body problem, a simplification of the general problem where one of the bodies is taken to have negligible or infinitesimal mass. The earliest and simplest particular solution was discovered by Euler in 1765. In Euler's solution, the three masses are collinear, positioned according to their masses. Then at every future time, the masses remain collinear, and the distances between them remain at the same ratio. This positions are the equilibrium points of the system and they lie on the line joining the primaries and are called collinear equilibrium points ($L_{1,2,3}$) (see Fig. 1). Later on, Lagrange in 1772 discovered another particular solution. In Lagrange's solution the three masses occur at the vertices of an equilateral triangle. The masses then follow elliptical orbits around their center of mass while remaining in an equilateral triangle formation, the negligible mass remain stable at this point in its orbit (see Fig. 2).These points are the equilibrium points and they are two called Lagrangian triangular equilibrium points ($L_{4,5}$).

These equilibrium points lie on the ξη-plane.

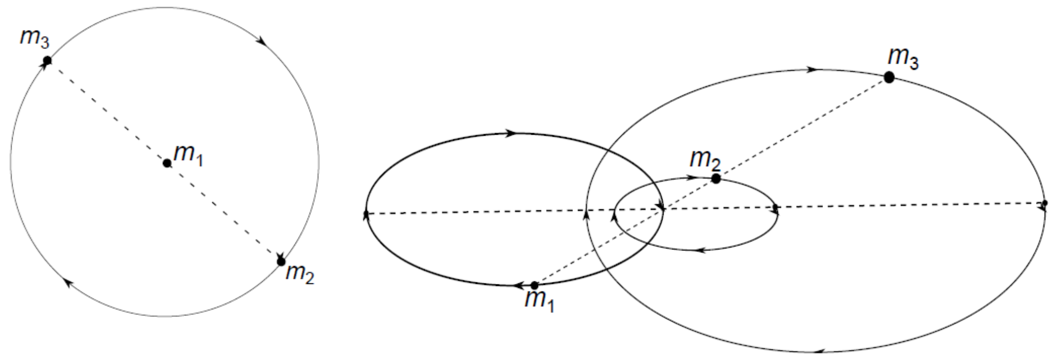

Fig. 1. The Euler solution: the three bodies remain collinear at all times, in elliptical orbits around the center of mass. Left: all masses equal. Right: unequal masses

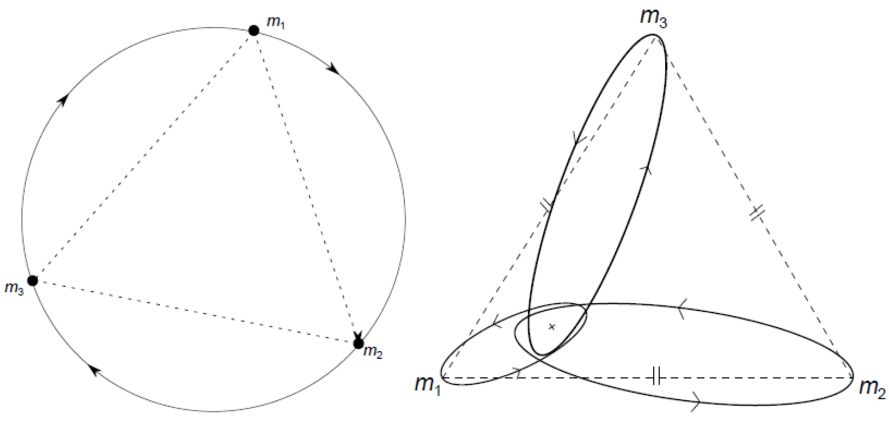

Fig. 2. The Lagrange solution: the three bodies form an equilateral triangle at all times. Left: three equal masses. Right: unequal masses

The existence of out-of-plane equilibrium points (OPEPs) in the ξζ- plane was discovered by Radzievskii [1,2] when studying the case of sun planet-particle and Galaxy-kernel-sun-particle and obtained the two equilibrium points $L_{6,7}$ on the ξζ- plane and is symmetrical with respect to the ξη-plane. Since then several authors singh [3-7] based their studies on the Radviesky model under different characterizations in Elliptic restricted three-body problem (ER3BP).

On the other hand [8-11] have studied the out-of-plane points in the ER3BP under the influence of radiation pressure or Pr-drag or oblateness or a combination of one of these forces and they found that the OPEPs is unstable.

Hussain and Umar [8] and [9] studied the OPEPs by considering the effect of the shapes (oblate and triaxial) of the primaries on the infinitesimal mass. [8] found that OPEPs are affected by the oblateness of the primary, radiation pressure and triaxiality of the secondary, semi-major axis, and eccentricity of the orbits of the principal bodies. But the OPEPs is unaffected by the semi-major axis and eccentricity of the orbits of the principal bodies. They found numerically that the binary system PSR 1903+0327 and DP-Leonis OPEPs is stable for low eccentricities.

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Numerical and graphical computation of out-of-plane equilibria by [10] for different values of the parameters (*μ, α, e, k* and *σ*) where *μ, α, e, k* and *σ* are mass parameter, albedo factor, eccentricity, ratio of the luminosity of smaller primary to luminosity of bigger primary considered as constant and oblateness factor due to smaller primary, respectively were used to analyse the effect of albedo on OPEPs and the equilibria are found to be unstable.

Singh and Richard [11] studies the motion of the out-of-plane equilibrium points within the framework of the Elliptic Restricted Three-Body Problem (ER3BP) at J4 of the smaller primary in the field of stellar binary systems: Xi- Bootis and Sirius around their common center of mass in elliptic orbits. He proved that the positions and stability of the out-of-plane equilibrium points are unstable in the lyapunov sense.

In addition to oblateness, triaxiality and radiation the study of OPEPs was extended to include the PR- drag force, one of the important component of radiation pressure force. The PR- Drag force is a component of radiation pressure and is tangential to the grain's motion. It is an effective force that opposes the direction of the dust grain's motion and causes a drop in the grain's angular momentum. In the studies above on photogravitational ER3BP this component of radiation force was ignored in the estimation of radiation pressure force. Authors like [12-15] have studied the impact of pr drag in comblnation with other prtubing forces.

In their paper [12] observed that OPEPs exist but incorporating the PR-drag results in non-zero y-coordinates of the OPEPs. Also, while studying the stability around the binary systems Luyten-726 and Sirius they found that at least two of the six roots of the characteristic equation, have positive real part and hence the OPEPs are unstable due to the presence of the PR- drag force. Using an analytical and numerical study, [13] shows that triangular equilibrium points exist in the plane of motion of the Sun-Earth system in the frame of the elliptic restricted problem of three bodies subject to the radial component of Poynting–Robertson (PR-drag) and radiation pressure factor of the bigger primary as well as dynamical flattening parameters of both primary bodies (i.e., Sun and Earth). However, triangular equilibrium points are linearly stable in the presence of the perturbing forces (including the PR-drag force) which shows that the perturbing forces have no significant effect on the positions of the triangular equilibrium points and their stability. Mishra and Bhola [14] examined the non-linear stability of triangular equilibrium points in the photogravitational elliptic restricted three body problem with Poynting-Robertson drag, where it was assumed the bigger primary is radiating and smaller primary an oblate spheroid. The condition of non-linear stability was estabilished using KAM theorem. They found three critical mass ratios and conclude that triangular equilibrium points are stable in the non-linear sense except at three critical mass ratios at which KAM theorem fails.

A study of the generalization of the Elliptic Restricted Three-Body Problem (ER3BP) by considering the effects of radiation (bigger primary) and oblate spheroid (smaller primary) by [15] shows that out of plane exist in the three dimensional case and the locations of $L_{6,7}$ are periodic and affected by A2 and radiation factor.

Interest in binary systems has increased, in the last decade, this is in part because many extra solar planetary systems revealed the presence of belts of dust particles that are regarded as the young analogues of Kuiper belt. [16] and [17] suggest that the position of the disc relative to the planets when they studied the effects of belts on planetary orbits and conclude that the planets might prefer to stay near the inner part instead of outer part of the belt. Later the R3BP was modified in their paper [18] to include the effect of additional gravitational force from the belt on the infinitesimal mass, which results in the formation of new libration (equilibrium) points.

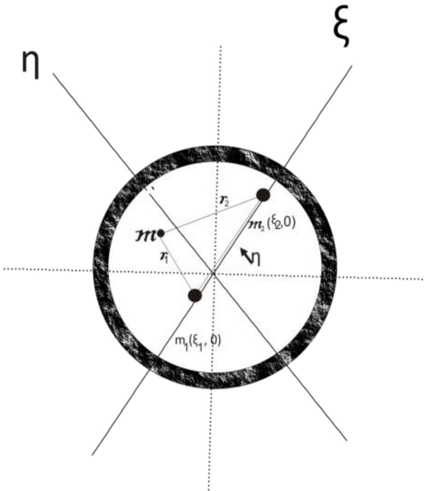

Image 1. The configuration of the problem

In the studies conducted on belt more attention is given to motion of the particle around triangular equilibrium points very few articles are available in OPEPs. The model by [19] focus on the CR3BP when the two primaries are oblate spheroids and radiating with the gravitational potential from a belt. They obtained in addition to the usual five libration points two new collinear points as a result of the potential from the belt. The influence of the belt and non-sphericity of the primaries on the infinitesimal mass was studied by [20]. They did analytic and numerical treatment of motion of a dust grain particle around triangular equilibrium points when the bigger primary is triaxial and the smaller one an oblate spheroid with a potential from the belt. They found that triangular points are stable for $0<\mu<\mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass ratio. It was also observed that the potential from the belt increase the range of stability.

In another study by [21] where the more massive primary is a triaxial body and less massive one an oblate spheroid emitting radiation enclosed by a circumbinary disc (belt) in the presence of Pr-drag force it was proved that the potential from the belt is a stabilizing force as it can change an unstable condition to a stable one even when the mass parameter exceeds the critical mass value $(\mu > \mu_c)$. Also, [9] found that the position and stability of out-of-plane equilibrium points are significantlyly affected by oblateness and radiation pressure of the primaries and the eccentricity of the orbits..Our work is an extension of [9] with radiating-triaxial primaries with a P-R drag force and a potential from the belt in the framework of ER3BP.This work to the best of our knowledge does not yet exist in the literature. The OPEPs has not yet been extensively researched, hence works devoted to it are few.

In this paper we investigate the effect of trixiality, radiation pressure with a P-R drag force and the potential of the belt on a test particle around the OPEPs in the framework of ER3BP.

This paper is organized in 6 sections. The first section is introduction, the equations of motion are described in section 2, locations of equilibrium points can be found in section 3, while section 4 contains the linear stability analysis of the out-of-plane equilibrium points using numerical applications, section 5 is discussion and finally section 6 is conclusion.

2. EQUATION OF MOTION

The equation of motion of an infinitesimal particle in the ER3BP when the primaries are triaxial and radiating with a P-R drag force and a gravitational potential from the belt, in a dimensionless rotating coordinate system *(ξ, η, ζ)* are as follows:

$$\xi'' - 2\eta' = \Omega_\xi$$
$$\eta'' + 2\xi' = \Omega_\eta$$
$$\zeta'' = \Omega_\zeta \qquad (1)$$

$$\Omega^\star = (1-e^2)^{-1/2}\left[\frac{1}{2}(\xi^2+\eta^2)+\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1}+\frac{(1-\mu)(2\sigma_1-\sigma_2)q_1}{2{r_1}^3}-\frac{3(1-\mu)(\sigma_1-\sigma_2)q_1\eta^2}{2r_1^5}-\frac{3(1-\mu)\sigma_1 q_1\zeta^2}{2{r_1}^5}+\frac{\mu q_2}{r_2}+\right.\right.$$

μ2σ3−σ4q22r23−3μσ3−σ4q2η22r25− 3μσ3q2ζ22r25+ Mbr2+c+ζ2+d2212 (2) *Ωξ=*

$$\frac{\partial\Omega^\star}{\partial\xi}-\frac{(1-e^2)^{-1/2}W_2F_1}{n^2r_2^2}$$

$$\Omega_\eta = \frac{\partial\Omega^\star}{\partial\eta} - \frac{(1-e^2)^{-1/2}W_2F_2}{n^2r_2^2}$$

where

$$\Omega_\zeta = \frac{\partial\Omega^\star}{\partial\zeta}$$

$$F_1 = \frac{(\xi+\mu)\{(\xi+\mu)\xi'+\eta\eta'+\zeta\zeta'\}+\xi'-n\eta}{{r_2}^2}$$

$$F_2 = \frac{\eta\{(\xi+\mu-1)\xi'+\eta\eta'+\zeta\zeta'\}+\eta'+n(\xi+\mu-1)}{{r_2}^2}$$

$$W_2 = \frac{\mu(1-q_2)}{c_d}$$

$${r_1}^2 = (\xi+\mu)^2+\eta^2+\zeta^2$$

$${r_2}^2 = = (\xi+\mu-1)^2+\eta^2+\zeta^2 \tag{3}$$

$$n^2 = \frac{1}{a}\left[1+\frac{3}{2}e^2+\frac{3}{2}(2\sigma_1-\sigma_2)+\frac{3}{2}(2\sigma_3-\sigma_4)+\frac{2M_br_c}{[r_c^2+T^2]^{3/2}}\right] \tag{4}$$

The effect of the gravitational potential of the belt is expressed using a model that explains a flattened potential and which best describes the gravitational potential within a system given by [22] as:

$$L(r,\zeta) = \frac{M_b}{\sqrt{r^2}+\left(c+\sqrt{\zeta^2+d^2}\right)} \tag{5}$$

r is the radial distance of the infinitesimal mass and is given by $r^2 = \xi^2+\zeta^2$, where c and d are the parameters which determine the density profile of the belt [22] and [23] r_c is the distance of any out-of-plane point from the origin and T is their sum, r_1 and r_2 are distances of the bigger and smaller primaries from the infinitesimal particle, respectively.q_1 and q_2 are their mass reduction factor (radiation factor),while (σ_1,σ_2) and (σ_3,σ_4) denote their triaxiality, respectively. n is the mean motion, a and e are the semi major axis and the eccentricity of the elliptic orbits respectively.

3. LOCATION OF OUT-OF-PLANE EQUILIBRIUM POINTS

The equilibrium points of the infinitesimal mass is obtained if the equation ξ′ = η′= ξ″ = η″ = ζ′= ζ″ = 0 satisfies Equation of motion (1); they are the solutions of the system of equations.

$\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$. Hence we have:

$$\Omega_\xi = \left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi+\mu)q_1}{{r_1}^3} + \frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)q_1}{2{r_1}^5} - \frac{15(1-\mu)(\xi+\mu)\sigma_1}{2{r_1}^7}q_1\eta^2 - \frac{15(1-\mu)(\xi+\mu)\sigma_1q_1\zeta^2}{2{r_1}^7} + \right.\right.$$

μξ+μ−1q2r23+ 3μξ+μ−12σ3−σ4q22r25−15μξ+μ−1σ32r27q2η2− 15μξ+μ−1σ3q2ζ22r27+Mb ξξ2+c+ζ2+d2232+W2ηnr22=0

(5)

$$\Omega_\eta = (1-e^2)^{-1/2}\,\eta\left[1-\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{{r_1}^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2{r_1}^5} + \frac{3(1-\mu)(\sigma_1-\sigma_2)}{{r_1}^5}q_1 - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2{r_1}^7}q_1\eta^2 - \right.\right.$$

151−μσ1q1ζ22r17+ μq2r23+ 3μ2σ3−σ4q22r25+3μσ3−σ4r25q2−15μσ3−σ42r27q2η2− 15μ σ3q2ζ22r27+Mbξ2+c+ζ2+d2232−W2(ξ+μ−1)nr22 = 0

(6)

$$\Omega_\zeta = (1-e^2)^{-1/2}\left[-\frac{\zeta}{n^2}\left\{\frac{(1-\mu)q_1}{r_1{}^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}q_1+\frac{3(1-\mu)\sigma_1}{r_1{}^5}q_1-\frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1{}^7}q_1\eta^2-\frac{15(1-\mu)\sigma_1 q_1\zeta^2}{2r_1{}^7}+\right.\right.$$

μq2r23+ *3μ2σ3−σ42r25q2+3μσ3r25q2−15μσ3−σ42r27q2η2−15μσ3q2ζ22r27+MbCζ2+d2−12+1ξ2+c+ζ2+d223/2* = 0 (7)

The out-of-plane equilibrium points are the solution of above equations, when

$\xi \neq 0$, $\eta = 0$ and $\zeta \neq 0$

From (7) with $\zeta \neq 0$ we get:

$$\frac{(1-\mu)q_1}{r_1{}^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}q_1+\frac{3(1-\mu)\sigma_1}{r_1{}^5}q_1-\frac{15(1-\mu)\sigma_1 q_1\zeta^2}{2r_1{}^7}+\frac{\mu q_2}{r_2{}^3}+\frac{3\mu(2\sigma_3-\sigma_4)}{2r_2{}^5}q_2+\frac{3\mu\sigma_3}{r_2{}^5}q_2-\frac{15\mu\sigma_3 q_2\zeta^2}{2r_2{}^7}+\frac{M_b\left[C\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{\frac{3}{2}}}=0 \quad (8)$$

Let $Q_1 = (1-\mu)q_1$ and $Q_2 = \mu q_2$, then (8) becomes

$$:\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{3Q_1\sigma_1}{r_1{}^5}-\frac{15Q_1\sigma_1\zeta^2}{2r_1{}^7}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{3Q_2\sigma_3}{r_2{}^5}-\frac{15Q_2\sigma_3\zeta^2}{2r_2{}^7}+\frac{M_b\left[C\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}=0 \quad (9)$$

Also from Equation (5) we write:

$$n^2\xi-\frac{Q_1(\xi+\mu)}{r_1{}^3}-\frac{3Q_1(\xi+\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{15Q_1(\xi+\mu)\sigma_1\zeta^2}{2r_1{}^7}-\frac{Q_2(\xi+\mu-1)}{r_2{}^3}-\frac{3Q_2(\xi+\mu-1)(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{15Q_2(\xi+\mu-1)\sigma_3\zeta^2}{2r_2{}^7}-\frac{M_{b\xi}}{\left[\xi^2+\left(c+\sqrt{\zeta^2+b^2}\right)^2\right]^{\frac{3}{2}}}+\frac{W_{2\eta}}{nr_2^2}=0 \quad (10)$$

Expanding Equation (10) we obtained:

$$\xi\left\{1-\frac{1}{n^2}\left(\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}-\frac{15Q_1\sigma_1\zeta^2}{2r_1{}^7}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}-\frac{15Q_2\sigma_3\zeta^2}{2r_2{}^7}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right)\right\}-\frac{\mu}{n^2}\left(\frac{Q_1}{r_1{}^3}-\frac{15Q_1\sigma_1\zeta^2}{2r_1{}^7}+\frac{Q_2}{r_2{}^3}-\frac{15Q_2\sigma_3\zeta^2}{2r_2{}^7}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}\right)+\frac{1}{n^2}\left(\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}-\right.$$

*15Q2σ3ζ22r27+W2*η*nr22=0*

(11)

From (9) we have:

$$\frac{15Q_1(\sigma_1-\sigma_2)\zeta^2}{2r_1{}^7}+\frac{15Q_2(\sigma_3-\sigma_4)\zeta^2}{2r_2{}^7}=\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{3Q_1\sigma_1}{r_1{}^5}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{3Q_2\sigma_3}{r_2{}^5}+\frac{M_b\left[c\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}$$

$$\zeta^2=\frac{2r_1{}^7 r_2{}^7}{15Q_1(\sigma_1-\sigma_2)r_2{}^7+15Q_2(2\sigma_3-\sigma_4)r_1{}^7}\left\{\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{3Q_1\sigma_1}{r_1{}^5}+\frac{3Q_2\sigma_3}{r_2{}^5}+\frac{M_b\left[c\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right\} \quad (12)$$

Substituting Equation (9) into Equation (11) and solving we obtained:

$$\xi\left\{1-\frac{1}{n^2}\left(-\frac{3Q_1\sigma_1}{{r_1}^5}-\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}-\frac{M_b\left[c+\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{\frac{3}{2}}}\right)\right\}$$
$$-\frac{\mu}{n^2}\left(-\frac{3Q_1\sigma_1}{{r_1}^5}-\frac{3Q_2\sigma_3}{{r_2}^5}-\frac{M_b\left[c+\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right)$$
$$+\frac{1}{n^2}\left(-\frac{Q_1}{{r_1}^3}-\frac{3Q_1(2\sigma_1-\sigma_2)}{2{r_1}^5}-\frac{3Q_1\sigma_1}{{r_1}^5}-\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{15Q_1\sigma_1\zeta^2}{2{r_1}^7}-\frac{M_b\left[c+\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right)+\frac{W_{2\eta}}{nr_2^2}=0$$

$$\text{i.e. } \xi=\frac{\frac{Q_1}{{r_1}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2{r_1}^5}+\frac{3Q_1\sigma_1(1-\mu)}{{r_1}^5}+\frac{3Q_2Q_1\sigma_3}{{r_2}^5}-\frac{15Q_1\sigma_1\zeta^2}{2{r_1}^7}+\frac{M_b\,Q_1\left[c+\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}+\frac{W_{2\eta}}{nr_2^2}}{n^2+\frac{3Q_1\sigma_1}{{r_1}^5}+\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}++\frac{M_b\left[c\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}}$$

Note that $Q_1 = 1-\mu$ and this is factored out from the above equation to obtain equation 13

$$\xi=\frac{(1-\mu)\left\{\frac{1}{{r_1}^3}+\frac{3(2\sigma_1-\sigma_2)}{2{r_1}^5}+\frac{3Q_1\sigma_1}{{r_1}^5}+\frac{3Q_2\sigma_3}{{r_2}^5}-\frac{15\sigma_1\zeta^2}{2{r_1}^7}+\frac{M_b\left[c+\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right\}+\frac{W_{2\eta}}{nr_2^2}}{n^2+\frac{3Q_1\sigma_1}{{r_1}^5}+\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}++\frac{M_b\left[c\left(\zeta^2+d^2\right)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}} \quad (13)$$

Equation 13 is ξ represents coordinate x.

We use the initial approximation $\xi_o = (1-\mu)$ and $\zeta_o = \sqrt{3(2\sigma_3-\sigma_4)}$ to obtain the positions of out-of-plane points $L_{6,7}$ numerically with the aid of the software package mathematica 10.4 in the form of power series to third order term in ($2\sigma_3$- σ_4) from Equation (12) and (13) as: (see [6] and[9]):

$$\xi_o = \frac{1}{2a\mu q_2}\left\{\left[(-1+\mu)-3\sqrt{3}(-1+\mu)(2+3e^2-2aq_1)\right]\right.$$
$$\left.+9W_2\left[(2\sigma_2-\sigma_1)(3-3aq_1)(2\sigma_3-\sigma_4)^{3/2}\right]\right\}$$
$$-\frac{1}{4a\mu q_2}\left\{\left[9W_2\sqrt{3}(-1+\mu)\left(2+3a\left(2+15(2\sigma_2-\sigma_1)q_1\right)(2\sigma_3-\sigma_4)^{5/2}\right]-\left[27W_2(-1+\mu)(2+\right.\right.$$
$$3e2-2aq1+2\sigma1-\sigma23-3aq1(-2-3e2+2\sigma2-\sigma1-3+6a$$
$$\left.\left.(-1+\mu)q_1)))(4(a^2\mu^2{q_2}^2)^{-1}(2\sigma_3-\sigma_4)^3+0\,(2\sigma_3-\sigma_4)^{7/2}\right]\right\} \quad (14)$$

$$\zeta_o=\sqrt{3}\sqrt{(2\sigma_3-\sigma_4)}-\frac{9(-1+\mu)(2+9(2\sigma_1-\sigma_2)q_1}{10\mu q_2}(2\sigma_3-\sigma_4)^2$$
$$+\frac{81(-1+\mu)}{20\mu q_2}(2+25)(2\sigma_1-\sigma_2)q_1(2\sigma_3-\sigma_4)^3-0\,(2\sigma_3-\sigma_4)^{7/2} \quad (15)$$

The equilibrium points $(\xi_0, 0, \pm\zeta_0)$ given by equations (14) and (15) are called the out-of-plane equilibrium points. This is the solution of the out of plane equilibrium points.

4. LINEAR STABILITY OF OUT-OF-PLANE EQUILIBRIUM POINTS

4.1 Variational Equation

Let the positions of any out-of-plane points be denoted by (ξ_0, η_0, ζ_0) and a small displacement from this position be denoted as (α, β, γ) having a new coordinates $(\xi_0 + \alpha, \eta_0 + \beta, \zeta_o + \gamma)$ in the neighbourhood of (ξ_0, η_0, ζ_0).

The variations can be written as:

$$\alpha = \xi - \xi_0, \beta = \eta - \eta_0, \gamma = \zeta - \zeta_o \tag{16}$$

It has the velocity

$$\dot{\xi} = \dot{\alpha}\ , \dot{\eta} = \dot{\beta}\ \dot{\zeta} = \dot{\gamma}$$

and acceleration:

$$\ddot{\xi} = \ddot{\alpha}\ \ddot{\eta} = \ddot{\beta}\ \ddot{\zeta} = \ddot{\gamma}$$

Substituting these values into Equation (1) and expanding the R.H.S. by Taylor series we obtained the variational equation as:

$$\begin{aligned} &\xi'' - 2\eta' == \alpha(\Omega_{\xi\xi})^0 + \beta(\Omega_{\xi\eta})^0 + \gamma(\Omega_{\xi\zeta})\ \dot{\alpha}(\Omega_{\xi\dot{\xi}})^0 + \dot{\beta}(\Omega_{\xi\dot{\eta}})^0 + \dot{\gamma}(\Omega_{\xi\dot{\zeta}})^0 \\ &\ddot{\beta} + 2\dot{\alpha} = \alpha(\Omega_{\eta\xi})^0 + \beta(\Omega_{\eta\eta})^0 + \gamma(\Omega_{\eta\zeta}) + \dot{\alpha}(\Omega_{\eta\dot{\xi}})^0 + \dot{\beta}(\Omega_{\eta\dot{\eta}})^0 + \dot{\gamma}(\Omega_{\eta\dot{\zeta}})^0 \\ &\ddot{\gamma} = \alpha(\Omega_{\zeta\xi})^0 + \beta(\Omega_{\zeta\eta})^0 + \gamma(\Omega_{\zeta\zeta}) + \dot{\alpha}(\Omega_{\zeta\dot{\xi}})^0 + \dot{\beta}(\Omega_{\zeta\dot{\eta}})^0 + \dot{\gamma}(\Omega_{\zeta\dot{\zeta}})^0 \end{aligned} \tag{17}$$

We consider only linear terms in α, β, γ. The second partial derivatives of Ω are denoted by subscript. The superscript 0,indicates that derivatives have been obtained at equilibrium point (ξ_0, η_0, ζ_0)

4.2 Characteristic Equation

Let the trial solution be as follows:

$$\begin{aligned} &\alpha = Pe^{\lambda t}\ \beta = Qe^{\lambda t}\ \gamma = Ve^{\lambda t} \\ &\dot{\alpha} = P\lambda e^{\lambda t}\ \dot{\beta} = Q\lambda e^{\lambda t}\ \dot{\gamma} = V\lambda e^{\lambda t} \\ &\ddot{\alpha} = P\lambda^2 e^{\lambda t}\ \ddot{\beta} = Q\lambda^2 e^{\lambda t}\ \ddot{\gamma} = V\lambda^2 e^{\lambda t} \end{aligned}$$

then substituting these values into variational Equation (17) we get:

$$\begin{aligned} &P\lambda^2 e^{\lambda t} - 2Q\lambda e^{\lambda t} = Pe^{\lambda t}\Omega^0{}_{\xi\xi} + Qe^{\lambda t}\Omega^0{}_{\xi\eta} + Ve^{\lambda t}\,\Omega_{\xi\zeta} + P\lambda e^{\lambda t}\Omega^0{}_{\xi\dot{\xi}} + Q\lambda e^{\lambda t}\,\Omega^0{}_{\xi\dot{\eta}} + V\lambda e^{\lambda t}\,\Omega_{\xi\dot{\zeta}} \\ &Q\lambda^2 e^{\lambda t} + 2P\lambda e^{\lambda t} = Pe^{\lambda t}\,\Omega^0{}_{\eta\xi} + Qe^{\lambda t}\,\Omega^0{}_{\eta\eta} + Ve^{\lambda t}\,\Omega^0{}_{\eta\zeta} + \lambda Pe^{\lambda t}\,\Omega^0{}_{\eta\dot{\xi}} + \lambda Qe^{\lambda t}\,\Omega^0{}_{\eta\dot{\eta}} + \\ &V\lambda e^{\lambda t}\,\Omega^0{}_{\eta\dot{\zeta}} \\ &V\lambda^2 e^{\lambda t} = Pe^{\lambda t}\,\Omega^0{}_{\zeta\xi} + Qe^{\lambda t}\,\Omega^0{}_{\zeta\eta} + Ve^{\lambda t}\,\Omega^0{}_{\zeta\zeta} + \lambda Pe^{\lambda t}\,\Omega^0{}_{\zeta\dot{\xi}} + \lambda Qe^{\lambda t}\,\Omega^0{}_{\zeta\dot{\eta}} + V\lambda e^{\lambda t}\,\Omega^0{}_{\zeta\dot{\zeta}} \end{aligned}$$

Multiplying through by $e^{-\lambda t}$ and rearranging we obtained the determinant as:

$$\begin{vmatrix} \lambda^2 - \Omega^0{}_{\xi\xi} - \lambda\Omega^0{}_{\xi\dot{\xi}} & -2\lambda - \Omega^0{}_{\xi\eta} + \lambda\,\Omega^0{}_{\xi\dot{\eta}} & -\Omega^0{}_{\xi\dot{\eta}} - \lambda\Omega_{\xi\dot{\zeta}} \\ 2\lambda - \Omega^0{}_{\eta\xi} - \lambda\,\Omega^0{}_{\eta\dot{\xi}} & \lambda^2 - \Omega^0{}_{\eta\eta} - \lambda\Omega^0{}_{\eta\dot{\eta}} & -\Omega^0{}_{\eta\zeta} - \lambda\Omega^0{}_{\eta\dot{\zeta}} \\ -\Omega^0{}_{\zeta\xi} - \lambda\Omega^0{}_{\zeta\dot{\xi}} & -\Omega^0{}_{\zeta\eta} - \lambda\Omega^0{}_{\zeta\dot{\eta}} & \lambda^2 - \Omega^0{}_{\zeta\zeta} - \lambda\,\Omega^0{}_{\zeta\dot{\zeta}} \end{vmatrix} = 0$$

Expanding the determinant we have:

$(\lambda^2 - \Omega^0{}_{\xi\,\xi} - \lambda\Omega^0{}_{\xi\,\dot{\xi}})[(\lambda^2 - \Omega^0{}_{\eta\eta} - \lambda\Omega^0{}_{\eta\dot{\eta}}$
$)(\lambda^2 - \Omega^0{}_{\zeta\zeta} - \lambda\,\Omega^0{}_{\zeta\dot{\zeta}}) - \{(-\Omega^0{}_{\zeta\eta} - \lambda\Omega^0{}_{\zeta\dot{\eta}})(-\Omega^0{}_{\eta\zeta} - \lambda\Omega^0{}_{\eta\dot{\zeta}})\} - (-2\lambda - \Omega^0{}_{\xi\eta} +$
λΩ0 ξη−2λ−Ω0 ξη+λΩ0 ξη λ2− Ω0 ζζ− λΩ0ζζ−−Ω0 ζξ −λΩ0 ζ ξ− Ω0 ηζ− λΩ0ηζ+ −Ω0
ξη−λΩξζ2λ−Ω0 η ξ−λΩ0 η ξ − Ω0 ζη−λΩ0 ζη−−Ω0 ζξ −λΩ0 ζ ξλ2−Ω0 ηη−λΩ0 ηη=0
(18)

Further expansion of Equation (18) yield the characteristic Equation:

$$\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \quad (19)$$

Where:

$a_1 = -\Omega^0{}_{\zeta\zeta} - \Omega^0{}_{\eta\dot{\eta}} - \Omega^0{}_{\xi\,\dot{\xi}}$
$a_2 = -\Omega^0{}_{\zeta\zeta} - \Omega^0{}_{\eta\eta} - \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\,\dot{\xi}} + \Omega^0{}_{\eta\dot{\eta}}\Omega^0{}_{\zeta\zeta} - \Omega^0{}_{\xi\,\xi} \mp \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\zeta\dot{\zeta}} + \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\eta\dot{\eta}} + 4 + 4\Omega^0{}_{\zeta\dot{\zeta}} -$
$2\Omega^0{}_{\eta\,\dot{\xi}} + 2\Omega^0{}_{\eta\,\dot{\xi}}\Omega^0{}_{\zeta\dot{\zeta}} + 4\,\Omega^0{}_{\xi\dot{\eta}} - 2\Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\zeta\dot{\zeta}} - \Omega^0{}_{\eta\,\dot{\xi}}\Omega^0{}_{\xi\dot{\eta}} + \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\,\dot{\xi}}\Omega^0{}_{\zeta\dot{\zeta}} - \Omega_{\xi\dot{\zeta}}\Omega^0{}_{\zeta\,\dot{\xi}}$
$a_3 = \Omega^0{}_{\eta\eta}\Omega^0{}_{\zeta\dot{\zeta}} - \Omega^0{}_{\zeta\eta}\Omega^0{}_{\eta\dot{\zeta}} - \Omega^0{}_{\zeta\dot{\eta}}\,\Omega^0{}_{\eta\zeta} - \Omega^0{}_{\zeta\dot{\eta}}\Omega^0{}_{\eta\dot{\zeta}} + \Omega^0{}_{\eta\dot{\eta}}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\zeta\dot{\zeta}} + \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\eta\dot{\eta}} +$
$\Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\eta\eta} - \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\eta\dot{\eta}}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\dot{\zeta}} - 2\Omega^0{}_{\eta\,\xi} + 2\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\dot{\zeta}}\lambda -$
$2\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\dot{\zeta}} + 2\Omega^0{}_{\xi\eta} - 2\Omega^0{}_{\xi\eta}\Omega^0{}_{\zeta\dot{\zeta}} - \Omega^0{}_{\eta\,\dot{\xi}}\Omega^0{}_{\xi\eta} + \Omega^0{}_{\xi\eta}\Omega^0{}_{\eta\,\dot{\xi}}\Omega^0{}_{\zeta\dot{\zeta}} - \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\,\xi} -$
$\Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\dot{\zeta}} - \Omega^0{}_{\xi\zeta}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\dot{\eta}}$
$a_4 = -\Omega^0{}_{\eta\eta}\Omega^0{}_{\zeta\zeta} - \Omega^0{}_{\zeta\eta}\,\Omega^0{}_{\eta\zeta} + \Omega^0{}_{\zeta\zeta}\Omega^0{}_{\xi\,\xi} + \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\eta\eta} - \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\eta\dot{\eta}}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\dot{\zeta}}$
$\quad - \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\eta\eta}\Omega^0{}_{\zeta\dot{\zeta}} - \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\eta\dot{\eta}}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\zeta\eta}\Omega^0{}_{\eta\dot{\zeta}} + \Omega^0{}_{\xi\,\dot{\xi}}\,\Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\zeta}$
$\quad + 2\,\Omega^0{}_{\eta\,\dot{\xi}}\Omega^0{}_{\zeta\zeta} - 2\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\dot{\zeta}} - 2\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\zeta} - \Omega^0{}_{\xi\eta}\Omega^0{}_{\eta\,\xi} - \Omega^0{}_{\xi\eta}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\dot{\zeta}}$
$\quad - 2\Omega^0{}_{\zeta\zeta}\Omega^0{}_{\xi\dot{\eta}} + \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\dot{\zeta}} + \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\,\dot{\xi}}\Omega^0{}_{\zeta\zeta} - \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\xi\zeta}\Omega^0{}_{\eta\dot{\zeta}}$
$\quad - \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\zeta} + 2\Omega^0{}_{\xi\zeta}\Omega^0{}_{\zeta\dot{\eta}} - \Omega^0{}_{\xi\zeta}\Omega^0{}_{\eta\,\dot{\xi}}\,\Omega^0{}_{\xi\dot{\eta}} - \Omega^0{}_{\xi\zeta}\Omega^0{}_{\zeta\xi}$
$\quad + \Omega^0{}_{\xi\zeta}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\dot{\eta}} + 2\Omega_{\xi\dot{\zeta}}\Omega^0{}_{\zeta\eta} - \Omega_{\xi\dot{\zeta}}\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\dot{\eta}} + \Omega_{\xi\dot{\zeta}}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\eta}$
$\quad + \Omega_{\xi\dot{\zeta}}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\eta}$
$a_5 = -\Omega^0{}_{\xi\,\xi}\Omega^0{}_{\eta\eta}\,\Omega^0{}_{\zeta\dot{\zeta}} - \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\eta\dot{\eta}}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\zeta\eta}\Omega^0{}_{\eta\dot{\zeta}} + \Omega^0{}_{\xi\,\xi}\,\Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\zeta} + \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\eta\eta}\Omega^0{}_{\zeta\zeta}$
$\quad + \Omega^0{}_{\xi\,\dot{\xi}}\Omega^0{}_{\zeta\eta}\,\Omega^0{}_{\eta\zeta} + 4\,\Omega^0{}_{\zeta\dot{\zeta}} + 2\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\zeta} - 2\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\dot{\zeta}} - 2\Omega^0{}_{\zeta\zeta}\Omega^0{}_{\xi\eta}$
$\quad + \Omega^0{}_{\xi\eta}\Omega^0{}_{\eta\,\xi}\,\Omega^0{}_{\zeta\dot{\zeta}} + \Omega^0{}_{\eta\,\dot{\xi}}\,\Omega^0{}_{\zeta\zeta}\Omega^0{}_{\xi\eta} - \Omega^0{}_{\xi\eta}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\dot{\zeta}} - \Omega^0{}_{\xi\eta}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\zeta}$
$\quad - \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\zeta} + 2\Omega^0{}_{\xi\zeta}\,\Omega^0{}_{\zeta\eta} - \Omega^0{}_{\xi\zeta}\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\xi\dot{\eta}} - \Omega^0{}_{\xi\zeta}\Omega^0{}_{\eta\,\dot{\xi}}\,\Omega^0{}_{\zeta\eta}$
$\quad + \Omega^0{}_{\xi\zeta}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\dot{\eta}} + \Omega^0{}_{\xi\zeta}\Omega^0{}_{\zeta\,\dot{\xi}}\Omega^0{}_{\eta\eta} - \Omega_{\xi\dot{\zeta}}\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\eta} + \Omega_{\xi\dot{\zeta}}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\eta}$
$a_6 = \Omega^0{}_{\xi\,\xi}\Omega^0{}_{\eta\eta}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\,\xi}\,\Omega^0{}_{\zeta\eta}\Omega^0{}_{\eta\zeta} + \Omega^0{}_{\xi\eta}\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\zeta} - \Omega^0{}_{\xi\eta}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\zeta} + \Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\zeta} +$
$\Omega^0{}_{\xi\dot{\eta}}\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\zeta} - \Omega^0{}_{\xi\zeta}\Omega^0{}_{\eta\,\xi}\Omega^0{}_{\zeta\eta} + \Omega^0{}_{\xi\zeta}\Omega^0{}_{\zeta\xi}\,\Omega^0{}_{\eta\eta} = 0$

The second partial derivatives evaluated at equilibrium point (ξ_0, η_0, ζ_0) as denoted by the superscript O are as follows:

$\Omega^0{}_{\xi\xi} = (1 - e^2)^{-1/2}\left[1 + \frac{1}{n^2}\left\{\frac{3Q_1(\xi_o+\mu)^2}{r_{10}{}^3} - \frac{Q_1}{r_{10}{}^3} + \frac{15Q_1(\xi_o+\mu)^2(2\sigma_1-\sigma_2)}{2r_{10}{}^7} - \frac{3Q_1}{2r_{10}{}^5} + \frac{105Q_1(\xi_o+\mu)^2\sigma_1\zeta_o{}^2}{2r_{10}{}^9} -\right.\right.$
15Q1σ1ζo22r107+ 3Q2ξo+μ−12r205−Q2r205+15Q2ξo+μ−122σ3−σ42r207−3Q22r205− 105Q2ξo+μ−12σ3ζo22r209+ 15Q2σ3ζo22r207+ 3Mbξ02ξ02+c+ζ02+d225/2− Mbξ02+c+ζ02+d223/2

$\Omega^0{}_{\eta\eta} = (1-e^2)^{-1/2}\Big[1 - \frac{1}{n^2}\Big\{\frac{Q_1}{{r_{10}}^3} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2{r_{10}}^5} - \frac{15Q_1\sigma_1{\zeta_o}^2}{2{r_{10}}^7} + \frac{Q_2}{{r_{20}}^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2{r_{20}}^5} - \frac{15Q_2\sigma_3{\zeta_o}^2}{2{r_{20}}^7} -$
3Mbη02ξ02+c+ζo2+d2252+Mbξ02+c+ζo2+d223/2

$\Omega^0{}_{\zeta\zeta} = (1-e^2)^{-1/2}\Big[\frac{1}{n^2}\Big\{-\frac{Q_1}{{r_{10}}^3} + \frac{3Q_1{\zeta_o}^2}{{r_{10}}^5} - \frac{3Q_1(2\sigma_1-\sigma_2)}{2{r_{10}}^5} + \frac{15Q_1(2\sigma_1-\sigma_2){\zeta_o}^2}{2{r_{10}}^7} - \frac{3Q_1\sigma_1}{{r_{10}}^5} + \frac{15Q_1\sigma_1{\zeta_o}^2}{{r_{10}}^7} + \frac{45Q_1\sigma_1{\zeta_o}^2}{2{r_{10}}^7} -$
105Q1σ1ζo42r109−Q2r203+3Q2ζo2r205−
3Q22σ3−σ42r205+15Q22σ3−σ4ζo22r207−3Q2σ3r205+15Q2σ3ζo22r207+45Q2σ3ζo22r207−10
5Q2σ3ζo42r209−Mbcζ02+d2−12+1ξ02+c+ζ02+d2232+Mbc2ζ02ζ02+d2−32ξ02+c+ζ02+d223
2+3Mbζ02cζ02+d2−12 +12ξ02+c+ζ02+d2252

$\Omega^0{}_{\xi\zeta} = (1-e^2)^{-1/2}\Big[\frac{3\zeta_0}{n^2}\Big\{\frac{Q_1(\xi_o+\mu)}{{r_{10}}^5} + \frac{5Q_1(\xi_o+\mu)(2\sigma_1-\sigma_2)}{2{r_{10}}^7} - \frac{35Q_1(\xi_o+\mu)\sigma_1{\zeta_o}^2}{2{r_{10}}^9} - \frac{15Q_1\sigma_1{\zeta_o}^2}{{r_{10}}^7} + \frac{Q_2(\xi_o+\mu-1)}{{r_{20}}^5} +$
5Q2ξo+μ−12σ3−σ42r207− 35Q2ξo+μ−1σ3ζo22r209+ 15Q2σ3ζo2r207+3Mbξ02 cζ02+d2−12
+1ξ02+c+ζ02+d225/2 (20)

5. NUMERICAL APPLICATION

We present the effect of triaxiality, belt and radiation pressure on the locations (Eqns.14 and 15) and stability (Equation 20) of OPEPs using arbitrary values In Tables 1- 4 , while in Tables 6-9 the effects on the binary system (xi-Bootis and Kruger 60) are shown. The results in Tables 6-9 were obtained by substituting the values of the orbital parameters (fixed) of the binary system (xi-Bootis and Kruger 60) and the varied values of triaxiality and radiation into (Equations 14 and 15) and Equation 20 for the locations and stability respectively.

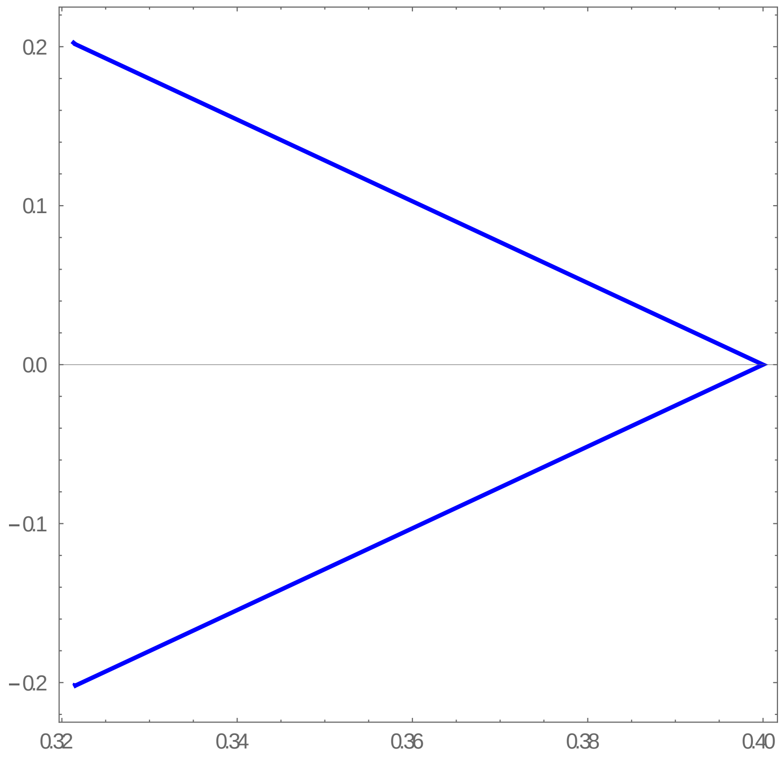

Fig. 3. Graph showing the effect of triaxiality on the OPEPs of XI-Bootis

Table 1. The effect of Triaxiality on the location and stability of out-of-plane equilibrium points for e =0.3, a =0.87, $\mu = 0.45$ q_1 =0.9988, q_2 =0.9977, M_b = 0.01, $W_2 = 1.17659 \times 10^{-12}$

S/no	Triaxiality				Out-of plane points		Roots of the characteristic equation		
	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.00	0.00	0.00	0.00	0.521012	0.311723	± 87.4420	120.2235	$\pm 33.9675i$
2.	0.02	0.015	0.003	0.002	0.477810	0.269001	± 87.5631	± 120.5631	$\pm 34.12654i$
3.	0.03	0.019	0.004	0.003	0.49061	0.24443	± 87.6615	± 120.9985	$\pm$34.241001i
4.	0.04	0.02	0.005	0.004	0.517438	0.219650	± 88.43423	± 121.43423	$\pm 35.35790i$
5	0.05	0.03	0.006	0.005	0.539144	0.209341	± 88.9780	± 122.110	± 35.41283

Table 2. The effect of belt on the location and stability of out-of-plane equilibrium points for e =0.3, a =0.87, $\mu = 0.45$ q_1 =0.9988, q_2 =0.9977, $W_2 = 1.17659 \times 10^{-12}$

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.01	0.06735	0.741593	−2.364473$\pm$0.364473i	$\pm$1.470823i	2.364473$\pm$0.364473i
2	0.02	0.04894	0.73965	−6.243416$\pm$0.765014i	$\pm$14.51723i	6.243416$\pm$0.765014i
3	0.03	0.03646	0.72671	−5.459825$\pm$0.886517i	$\pm$13.44601i	5.459825$\pm$0.886517i
4	0.04	0.03238	0.72136	−3.556463$\pm$0.876321i	$\pm$11.52649i	3.556463$\pm$0.876321i
5	0.05	0.02671	0.71641	−1.524192$\pm$0.837649i	$\pm$8.875206i	1.524192$\pm$0.837649i

Table 3. The Effect of radiaion pressure on the location and stability of out-of-plane equilibrium points for e=0.3,a=0.87,$\mu = 0.35$, $\sigma_1 = 0.02, \sigma_2 = 0.015, \sigma_3 = 0.003, \sigma_4 = 0.004$, M_b= 0.01, $W_2 = 1.17659 \times 10^{-12}$

S/ no	Radiaion pressure		Out-of plane points		Roots of the characteristic equation		
	q_1	q_2	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.9960	0.9950	0.66735	0.412681	−2.54373$\pm$0.543728i	$\pm$11.42462i	2.54373$\pm$0.543728i
2	0.9964	0.9954	0.67024	0.394326	−3.24342$\pm$ 0.810034i	$\pm$16.23703i	3.24342$\pm$0.810034i
3	0.9968	0.9958	0.67646	0.343671	−6.45986$\pm$0.886517i	$\pm$27.42462i	6.45986$\pm$0.886517i
4	0.9972	0.9962	0.68434	0.328763	−10.5756$\pm$0.47632i	$\pm$38.57823i	10.5756$\pm$0.47632i
5	0.9976	0.9966	0.69101	0.310641	−13.4140$\pm$0.357649i	$\pm$45.41365i	13.4145$\pm$0.357649i

Table 4. The Combined effect of the pertubations on the location and stability of out-of-plane equilibrium points for e = 0.3,a = 0.34 $W_2 = 1.17659 \times 10^{-12}$

(a)

S/no.	Triaxiality				Radiation factors		Belt	Mass ratio
	σ_1	σ_2	σ_3	σ_4	q_1	q_2	M_b	μ
1.	0.02	0.01	0.002	0.001	0.9980	0.9976	0.01	0.0375
2.	0.03	0.02	0.003	0.002	0.9984	0.9980	0.02	0.0380
3.	0.04	0.03	0.004	0.003	0.9988	0.9984	0.03	0.0385
4.	0.05	0.04	0.005	0.004	0.9992	0.9988	0.04	0.0390
5.	0.06	0.05	0.006	0.005	0.9996	0.9992	0.05	0.0395

(b)

out-of-plane points		The characteristic roots		
ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.633412	0.197065	± 227.584	± 19.2802	$\pm 86.7122i$
0.633011	0.205634	± 331.385	± 90.4429	$\pm 87.2517i$
0.632785	0.218767	± 379.863	± 90.547	$\pm$88.5313i
0.632145	0.224261	± 463.07	± 90.5989	$\pm 88.9132i$
0.631004	0.234659	± 88.9780	± 122.110	$\pm 35.41283i$

Table 5. Numerical data for the Binary System

Binary system	Masses (MΘ)		Eccentricity (e)	Semi-major axis (a)	Luminosity LΘ		Spectral types
	M_1	M_2			L_1	L_2	
Xi Bootis	0.9	0.66	0.5117	4.9044	0.49	0.061	G8/k4
Kruger 60	0.271	0.176	0.4100	2.3830	0.01	0.0034	M3/M4

Source: NASA ADS

Table 6. The effect of triaxiality on the location and stability of out-of-plane equilibrium points of xi-Bootis for e = 0.5117, a = 0.7304, μ = 0.4231 q_1 = 0.9988, q_2 = 0.9998, $W_2 = 1.17659 \times 10^{-12}$

S/no	Triaxiality				Out-of plane points		Roots of the characteristic equation		
	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.015	0.011	0.002	0.001	0.466010	0.275418	$\pm$610.524	-173.012$\pm$ 184.316i	173.012$\pm$ 184.316i
2.	0.02	0.015	0.003	0.002	0.477810	0.269001	$\pm$814.061	-175.981$\pm$ 182.895i	175.981$\pm$ 182.895i
3.	0.03	0.019	0.004	0.003	0.49061	0.24443	$\pm$998.23	-174.887$\pm$ 180.49i	174.887$\pm$ 180.49i
4.	0.04	0.02	0.005	0.004	0.517438	0.219650	$\pm$1627.48	-175.13$\pm$ 176.832i	175.13$\pm$ 176.832i
5.	0.05	0.03	0.006	0.005	0.539144	0.209341	$\pm$1321.43	-173.39$\pm$ 176.972i	173.39$\pm$ 176.972i

Table 7. The effect of belt (M_b) on the location and stability of out-of-plane equilibrium points of xi-Bootis for e = 0.5117, a = 0.7304, μ = 0.4231 q_1 = 0.9988, q_2 = 0.9998. σ_1=0.02, σ_2=0.015, σ_3=0.003, σ_4=0.002, $W_2 = 1.17659 \times 10^{-12}$

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.02	0.521012	0.211723	$\pm$ 47.347306	$\pm113.5678\,i$	±33.98450
2	0.03	0.513422	0.211965	$\pm$47.748921	$\pm120.5631i$	±34.12654
3	0.04	0.51200	0.221343	$\pm$48.256439	$\pm120.9985i$	±34.241001
4	0.05	0.51042	0.229867	$\pm$48.84320	$\pm121.43423i$	±35.35790
5	0.06	0.50964	0.239341	$\pm$ 49.22418	$\pm122.1101i$	±35.4128321

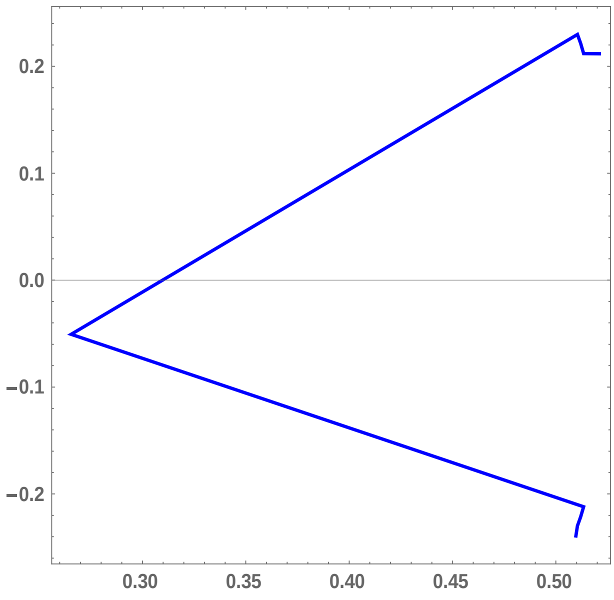

Fig. 4. Graph showing the effect of the belt on the OPEPs of XI-Bootis

Table 8. The effect of triaxiality on the location and stability of out-of-plane equilibrium points of Kruger 60 for e = 0.4100, a = 0.5894, μ = 0.3937 q_1= 0.9992 and q_2 = 0. 9996, $W_2 = 1.17659 \times 10-12$

S/No	Triaxiality				Out-of-plane points		Roots of the characteristic equation		
	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.02	0.002	0.002	0.001	0.946710	0.241070	-37.2193$\pm$ 21.4739i	0$\pm$42.9477i	37.2193$\pm$21. 47396i
2.	0.03	0.025	0.003	0.002	0.951193	0.216110	-38.567$\pm$ -22.0922i	0$\pm$44.1875i	38.567$\pm$ 22.0922i
3.	0.04	0.035	0.004	0.003	0.958414	0.213279	-51.476$\pm$ 28.7198i	0$\pm$57.5122i	51.476$\pm$ 28.7198i
4.	0.05	0.045	0.005	0.004	0.959130	0.207454	-100.461$\pm$ 55.2628i	0$\pm$110.803i	100.461$\pm$ 55.2628i
5.	0.06	0.055	0.006	0.005	0.960314	0.204511	-154.48$\pm$ 81.4207i	0$\pm$164.244i	154.48$\pm$ 81.4207i

Table 9. The effect of belt (M_b) on the location and stability of out-of-plane equilibrium points of Kruger-60 for e = 0.4100, a = 0.5894, μ = 0.3937 q_1= 0.9992 and q_2 = 0. 9996 . σ_1=0.02, σ_2=0.015, σ_3=0.003, σ_4=0.002, $W_2 = 1.17659 \times 10^{-12}$

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.01	0.321012	0.200534	$\pm$ 54.223624	$\pm$19.2802	$\pm$86.7122i
2	0.02	0.321342	0.201823	$\pm$ 54.534534	$\pm$90.4429	$\pm$87.2517i
3	0.03	0.321440	0.201944	$\pm$54.655978	$\pm$90.547	$\pm$88.5313i
4	0.04	0.321452	0.202112	$\pm$55.232720	$\pm$90.5989	$\pm$88.9132i
5	0.05	0.3214634	0.202472	$\pm$ 55.703529	$\pm$122.110	$\pm$35.41283i

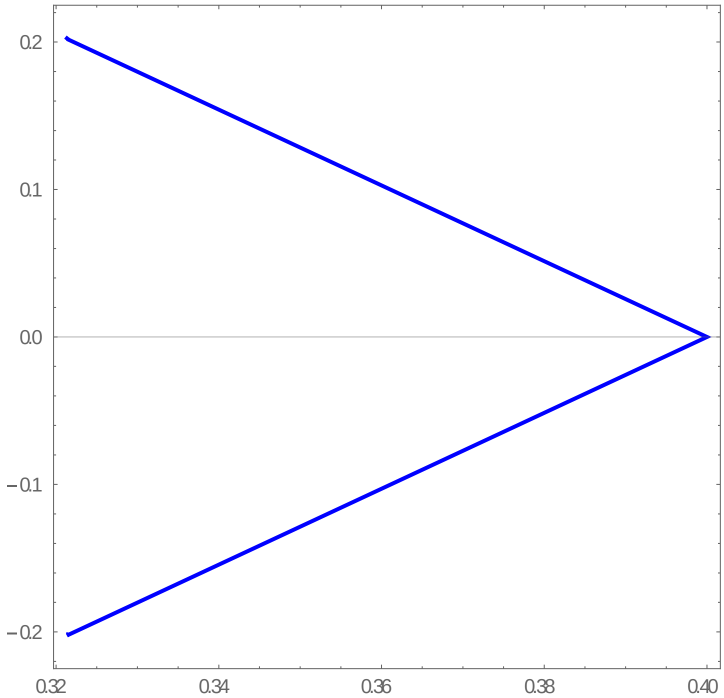

Fig. 5. Graph showing the effect of triaxiality on the OPEPs of Kruger-60

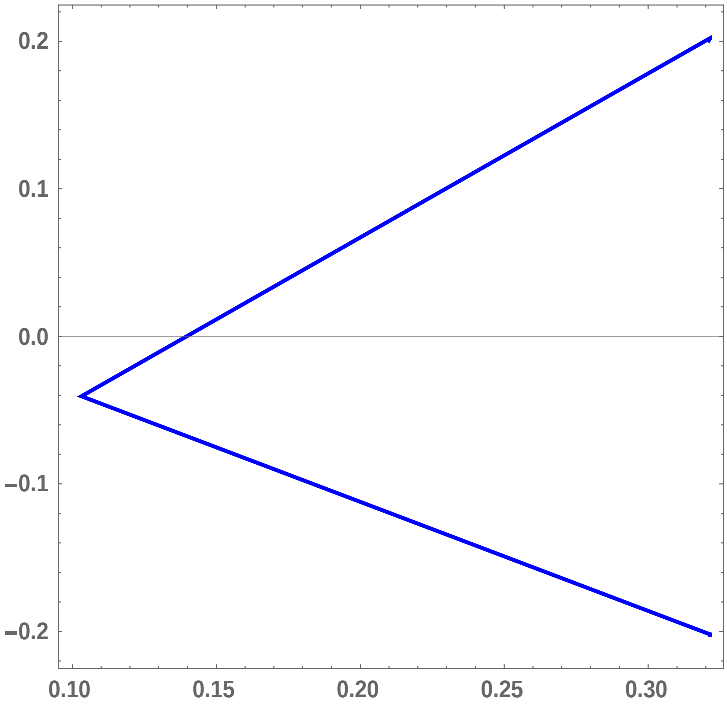

Fig. 6. Graph showing the effect of the belt on the OPEPs of Kruger-60

In Table 5 we present the numerical data of the binary system xi-Bootis and Kruger 60.

6. DISCUSSION

The motion of a third body under the influence of triaxial and radiating primaries together with a circumbinary disc and a P-R drag force has been described in equation (1)-(4). The positions of out-of-plane equilibrium points are given in equations 14 and 15 and are first obtained analytically and then numerically by power series expansion about the triaxiality coefficient of the smaller primary in Equations 14 and 15 to third order term with the aid of the software MATHEMATICA 10.4.The stability of these points are obtained by solving the roots of Equation (19) numerically. The positions of out-of-plane points and the characteristic roots obtained using arbitrary values for the parameters are shown in Tables 1-4 and Table 6-9. The arbitrary values for the parameters were substituted into Equation (19) and solved numerically with MATHEMATICA 10.4.to get the roots seen in the Tables which are the eigenvalues of the characteristic Eqn. (19).

We can see from the tables that the roots are either complex with both positive and negative real parts or purely real therefore the OPEPs are unstable. According to [24], EPs are stable only if the six roots λ_i (i=1,2,3,4,5,6) are purely imaginary roots or complex roots with negative real parts and are unstable if λ_i (i=1,2,3,4,5,6) are complex or real roots.

Tables 1 and 2, shows that the point $L_{6,7}$ shifts towards the line joining the primaries as the effects of triaxiality and belt are being increased respectively, while in Table 3 $L_{6,7}$ is seen to move away from the line joining the primaries as the radiation factors were increased. The combined effects of all the parameters are shown in Table 4.The arbitrary values assign to the parameters are shown in Table 4a. Table 4b shows their effects on OPEPs and its stability, In all cases the out-of- plane equilibrium points moves away from the ξ-axis when the values of the parameters were increased. The roots (λ_i (i=1,2,3,4,5,6)) in Tables 1-4 are complex or real roots,hence the OPEPs are unstable. The numerical data of the binary systems (xi- bootis and kruger-60) are shown in Table 5. The effects of triaxiality and the belt on the binary systems can be observed in Tables 6-9 and Figs. 3-6. These Tables and the graphs shows that increasing the values of triaxiality and belt, while keeping the orbital parametres of the Xi-bootis and Kruger-60 constant, results in a shift of the OPEPs. It can be seen in Table 6 that OPEPs shifts towards the ξ-axis this can be seen clearly in Fig. 3,this in contrast to the effect of the belt on

OPEPs of Xi-bootis in Table 7, where OPEPs shifts away from the the line joining the primaries (see also Fig. 4). The OPEPs in both Tables are unstable due to nature of their roots which are complex or real roots. The effects of triaxiality and the belt on Kruger-60 is similar to their effects on Xi-bootis. The effects of triaxiality moves the OPEPs towards the line joining the primaries (see Table 8 and Fig. 5), while the effect of the belt moves OPEPs away from the ξ-axis (See Table 9 and Fig. 6).Similar to what we obtained in the case of xi-Bootis, the roots obtained for OPEPs of Kruger-60 are either complex or real as such OPEPs are unstable. The changes in the positions of OPEPs are as shown in the graphs (Figs. 2-6). The instability OPEPs have been confirmed by [6], and [9].

7. CONCLUSION

We have obtained the out of plane equilibrium points and their stability in the framework of ER3BP when the primaries are triaxial,radiating with P-R drag force and surrounded by a belt.It is found that the positions are affected by triaxiality, radiation and the belt.We found that for the binary systems the effect of triaxialty and the belt moves OPEPs in opposite directions-while the effect of triaxiality moves OPEPs towards the ξ-axis,the belt moves OPEPs away from the ξ-axis. It was also observed that using arbitrary values and the values of the binary systems the OPEPs still remain unstable.Our OPEPs (Equations 14 and 15) tally with that of [9] when $(2\sigma_1 - \sigma_2) = A_1$ and $(2\sigma_3 - \sigma_4) = A_2$.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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