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Generalized (α, β)-higher Derivations on Lie Ideals of Rings with Involution

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this manuscript, we investigate the behaviour of additive mappings which satisfy a functional identity associated with generalized (α, β)-higher derivations on Lie ideals of a prime ring with involution. As the consequences of our main theorem, many well known results can be deduced.

Keywords: Involution; $*$-closed Lie ideal; (α, β)-higher derivation; generalized (α, β)-higher derivation.

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1 Introduction

Hasse and Schmidt [1] were who extended the concept of derivations to higher derivations. More precisely, they supplied that $D = \{d_n\}_{n\in\mathbb{N}}$, a family of additive mappings on R, is said to be a higher derivation (resp. Jordan higher derivation) on R if $d_0 = I_R$ and $d_n(xy) = \sum\limits_{i+j=n} d_i(x)d_j(y)$ (resp. $d_n(x^2) = \sum\limits_{i+j=n} d_i(x)d_j(x)$) for all $x, y \in R$ and for each $n \in \mathbb{N}$. It is easy to see that the first member of each higher derivation is itself a derivation. More related result can be find in [2]. Later

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on, Cortes and Haetinger [3] defined generalized higher derivations. A family $F = (f_i)_{i\in\mathbb{N}}$ of additive mapping of a ring R such that $f_0 = I_R$ is said to be a generalized higher derivation (resp. generalized Jordan higher derivation) of R if there exists a higher derivation (resp. Jordan higher derivation) $D = \{d_n\}_{n\in\mathbb{N}}$ and for each $n \in \mathbb{N}$, $f_n(xy) = \sum\limits_{i+j=n} f_i(x)d_j(y)$ (resp. $f_n(x^2) = \sum\limits_{i+j=n} f_i(x)d_j(x)$) holds for all $x, y \in R$. Obviously, every generalized higher derivation is a generalized Jordan higher derivation but the converse need not to be true. The converse have been proved in [3] for square closed Lie ideals of a prime ring. Later, Wei and Xao [4] established this result for a 2-torsion free semiprime ring. For an account on higher derivations, we refer the reader to [5, 6, 7]. In 2010, Ashraf et al [6] introduced the concept of (θ, ϕ)-higher derivations as follows: A family D of additive mappings d_n on R is said to be a (θ, ϕ)-higher derivation of R if $d_0 = I_R$ and $d_n(xy) = \sum\limits_{i+j=n} d_i(\theta^{n-i}(x))d_j(\phi^{n-j}(y))$ for all $x, y \in R$ and for each $n \in \mathbb{N}$. Further, in [5], Ashraf and Khan have acquanted the concept of generalized (θ, ϕ)-higher derivations and proved that every generalized Jordan (θ, ϕ)-higher derivations is a generalized (θ, ϕ)-higher derivation on Lie ideals of a prime ring R.

In [8], Herstein proved that if a simple ring with $char(R) \neq 2$ and $dim \geq 4$ admitting additive map $d : R \longrightarrow R$ such that $d(xx^*) = d(x)x^* + xd(x)$ for all $x \in R$, then d must be a derivation. Later Daif and El-Sayiad [9] obtained the following result: Let R be a 2-torsion free semiprime $*$-ring and $F : R \longrightarrow R$ be an additive mapping associated with a derivation d on R such that $F(xx^*) = F(x)x^* + xd(x^*)$ holds for all $x \in R$. Then F is a generalized Jordan derivation. In the same manner in [10], Ashraf et al. established Daif and El-Sayiad result in more general form and proved the following: Let R be a 2-torsion free semiprime $*$-ring. Suppose that α, β are endomorphisms of R such that α is an automorphism of R. If there exists an additive mapping $F : R \longrightarrow R$ associated with a (α, β)-derivation d of R such that $F(xx^*) = F(x)\alpha(x^*) + \beta(x)d(x^*)$ holds for all $x \in R$, then $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$. In 2015, Ezzat [14] have studied afforementioned results on generalized higher derivations.

Very recently, Husain et al. [11] extended Ezzat result for generalized (θ, ϕ)-higher derivations on a semiprime ring with involution. In the present paper, we study above mentioned theorem in the setting of generalized (θ, ϕ)-higher derivations on Lie ideals of a prime ring with involution.

2 Preliminaries and Main Result

Throughout this article, unless otherwise mentioned, R will denote an associative ring. A ring endowed with involution $*$ is called a ring with involution or $*$-ring. For basic definitions and notations we refer the reader to [12] and [8]. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$ and $r \in R$. U is also called $*$-closed if $U^* = U$.

We begin our discussion with following key lemma which will be extensively used to prove the main result.

Lemma 2.1. *[11, Lemma 1.2] Let R be a prime ring with involution with $char(R) \neq 2$ and U be a non central $*$-closed Lie ideal of R such that $u^2 \in U$, for all $u \in U$. Suppose that α is an automorphism of U. If there exists an element $a \in U$ such that $a\alpha(u^*) = a\alpha(u)$ holds for all $u \in U$, then $a \in Z(R)$.*

The main result of the present paper is the following theorem.

Theorem 2.2. *Let R be a prime ring of characteristic different from two with involution and U be a square close Lie ideal on R. Suppose there exists a family of additive mappings $F = (f_i)_{i\in N_0}$ of U into R associated with some (α, β)-higher derivation $D = (d_i)_{i\in N_0}$ of U into R, where α, β are* *commuting automorphisms on R, such that* $f_n(uu^*) = \sum_{i+j=n} f_i(\alpha^{n-i}(u))d_j(\beta^{n-j}(u^*))$ *is fulfilled for each* $n \in N$ *and for all* $u \in U$*. Then,* F *is a generalized* (α, β)*-higher derivation on* U *into* R*.*

Proof. Our hypothesis is

$$f_n(uu^*) = \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(u^*)) \tag{2.1}$$

for all $u \in U$. Linearization of above expression yields

$$\begin{aligned} f_n(uv^* + vu^*) \quad &= \quad \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(v^*)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v))d_j(\alpha^{n-j}(u^*)) \end{aligned} \tag{2.2}$$

for all $u, v \in U$. Taking $v = u^*$, we get

$$\eta_n(u) + \eta_n(u^*) = 0 \tag{2.3}$$

for all $u \in U$, where $\eta_n(u)$ stands for $f_n(u^2) - \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(u))$. Our aim is to show $\eta_n(u) = 0$ for all $u \in U$. To show $\eta_n(u) = 0$, we prosecute by induction. If $n = 0$, then it is easy to obtain $\eta_0(u) = 0$ for all $u \in U$. If $n = 1$, then one can easily prove the result by following [13, Theorem 2.3]. Now suppose $\eta_m(u) = 0$ for all $u \in U$, and for $m < n$. Set $A = 2(f_n(u(uv^* + vu^*) + (uv^* + vu^*)u^*))$. In view of (2.2), we have

$$\begin{aligned} A \quad &= \quad 2\Big(\sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(uv^* + vu^*)) + f_i(\beta^{n-i}(uv^* + vu^*))d_j(\alpha^{n-j}(u^*))\Big) \\ &= \quad 2\Big(\sum_{i+j=n} f_i(\beta^{n-i}(u))\Big(\sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{j-q}\alpha^{n-j}(v^*)) \\ &\quad + \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(v))d_q(\alpha^{j-q}\alpha^{n-j}(u^*))\Big) \\ &\quad + \sum_{i+j=n} \Big(\sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(u))d_t(\alpha^{i-t}\beta^{n-i}(v^*)) \\ &\quad + \sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(v))d_t(\alpha^{i-t}\beta^{n-i}(u^*))\Big)d_j(\alpha^{n-j}(u^*))\Big) \\ &= \quad 2\Big(\sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{j-q}\alpha^{n-j}(v^*)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(v))d_q(\alpha^{j-q}\alpha^{n-j}(u^*)) \\ &\quad + \sum_{i+j=n} \sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(u))d_t(\alpha^{i-t}\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^*)) \\ &\quad + \sum_{i+j=n} \sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(v))d_t(\alpha^{i-t}\beta^{n-i}(u^*))d_j(\alpha^{n-j}(u^*))\Big) \end{aligned}$$

This can be written as

$$\begin{aligned} A &= 2(\sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{n-q}(v^*)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(v))d_q(\alpha^{n-q}(u^*)) \\ &\quad + \sum_{i+j=n}\sum_{s+t=i} f_s(\beta^{n-s}(u))d_t(\alpha^{i-t}\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^*)) \\ &\quad + \sum_{i+j=n}\sum_{s+t=i} f_s(\beta^{n-s}(v))d_t(\alpha^{i-t}\beta^{n-i}(u^*))d_j(\alpha^{n-j}(u^*))) \end{aligned}$$

In particular, we have

$$\begin{aligned} A &= 2(\sum_{i+p=n} f_i(\beta^{n-i}(u))d_p(\alpha^{n-p}(u))\alpha^n(v^*) \\ &\quad + \sum_{\substack{i+p+q=n \\ i+p\neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q\alpha^i(u))d_q(\alpha^{n-q}(v^*)) \\ &\quad + \sum_{i+j+k=n} f_i(\beta^{n-i}(u))d_j(\beta^k\alpha^i(v+v^*))d_k(\alpha^{n-k}(u^*)) \\ &\quad + \sum_{i+j=n}\sum_{s+t=i} f_s(\beta^{n-s}(v))d_t(\alpha^{i-t}\beta^{n-i}(u^*))d_j(\alpha^{n-j}(u^*))) \end{aligned} \tag{2.4}$$

On the other hand, A can be written as

$$\begin{aligned} A &= 2(f_n(u(v+v^*)u^*) + f_n(u^2v^* + v(u^*)^2)) \\ &= 2(f_n(u(v+v^*)u^*) + \sum_{i+j=n} f_i(\beta^{n-i}(u^2))d_j(\alpha^{n-j}(v^*)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v))d_j(\alpha^{n-j}(u^*)^2)) \end{aligned}$$

$$\begin{aligned} A &= 2(f_n(u(v+v^*)u^*) + f_n(u^2)\alpha^n(v^*) \\ &\quad + \sum_{\substack{i+p+q=n \\ i+p\neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q\alpha^i(u))d_q(\alpha^{n-q}(v^*)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v)) \sum_{k+l=j} d_k(\alpha^{j-k}\beta^{n-j}(u^*))d_l\alpha^{n-l}(u^*)) \end{aligned} \tag{2.5}$$

Compare (2.4) and (2.5) and use the fact that the characteristic of R is different from two, we get

$$f_n(u(v+v^*)u^*) = -\eta_n(u)\alpha^n(v^*) + \sum_{i+j+k=n} f_i(\beta^{n-i}(u))d_j(\beta^k\alpha^i(v+v^*))d_k(\alpha^{n-k}(u^*))$$

Taking v as $v - v^*$ gives

$$\eta_n(u)\alpha^n(v^*) = \eta_n(u)\alpha^n(v) \tag{2.6}$$

In view of Lemma 2.1, we get $\eta_n(u) \in Z(R)$ for all $u \in U$.

Next, putting v as v^* in (2.2), we obtain

$$\begin{aligned} f_n(uv + v^*u^*) &= \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(v)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^*)) \end{aligned} \tag{2.7}$$

Replacing v by $2uv$ in above expression and using the fact that the characteristic of R is different from two, we get

$$\begin{aligned} f_n(u^2v+v^*u^{*2}) &= \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(uv)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*)) \\ &= \sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{j-q}\alpha^{n-j}(v)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*)) \\ &= \sum_{i+p=n} f_i(\beta^{n-i}(u))d_p(\alpha^{n-p}(u))\alpha^n(v) \\ &\quad + \sum_{\substack{i+p+q=n\\ i+p\neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q\alpha^i(u))d_q(\alpha^{n-q}(v))) \\ &\quad + f_n(v^*u^*)\alpha^n(u^*) + \sum_{\substack{i+j=n\\ i\neq n}} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*)). \end{aligned}$$

On the other hand, replacement of u by u^2 in (2.7) gives

$$\begin{aligned} f_n(u^2v+v^*u^{*2}) &= \sum_{i+j=n} f_i(\beta^{n-i}(u^2))d_j(\alpha^{n-j}(v)) + \sum_{i+j=n} f_i(\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^{*2})) \\ &= f_n(u^2)\alpha^n(v) + \sum_{\substack{i+p+q=n\\ i\neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q\alpha^i(u))d_q(\alpha^{n-q}(v)) \\ &\quad + \sum_{\substack{s+t+j=n\\ s+t\neq n}} f_s(\beta^{n-s}(v^*))d_t(\alpha^j\beta^s(u^*))d_j\alpha^{n-j}(u^*) \\ &\quad + \sum_{s+t=n} f_s(\beta^{n-s}(v^*))d_t(\alpha^{n-t}(u^*))\alpha^n(u^*) \end{aligned}$$

In view of the last two expressions, we have

$$\begin{aligned} 0 &= \eta_n(u)\alpha^n(v) + \Big(-f_n(v^*u^*) + \sum_{s+t=n} f_s(\beta^{n-s}(v^*))d_t(\alpha^{n-t}(u^*))\Big)\alpha^n(U^*) \\ &- \sum_{\substack{s+t+j=n\\ s+t\neq n}} f_s(\beta^{n-s}(v^*))d_t(\alpha^j\beta^s(u^*))d_j\alpha^{n-j}(u^*) - \sum_{\substack{i+j=n\\ i\neq n}} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*)) \end{aligned}$$

In particular for, $v=u$ gives

$$\eta_n(u)\alpha^n(u) - \eta_n(u^*)\alpha^n(u^*) = 0 \tag{2.8}$$

for all $u \in U$. From (2.3), we have

$$\eta_n(u)\alpha^n(u) + \eta_n(u)\alpha^n(u^*) = 0 \tag{2.9}$$

for all $u \in U$. Replacing v by u in (2.6), we obtain

$$\eta_n(u)\alpha^n(u) - \eta_n(u)\alpha^n(u^*) = 0 \tag{2.10}$$

for all $u \in U$. Combining above expression with (2.9), we get

$$\eta_n(u)\alpha^n(u) = 0 \tag{2.11}$$

for all $u \in U$. Linearization of (2.11) gives

$$\eta_n(u)\alpha^n(v) + \mu(u,v)\alpha^n(u) + \eta_n(v)\alpha^n(u) + \mu(u,v)\alpha^n(v) = 0$$

for all $u \in U$, where $\mu(u,v) = f_n(uv+vu) - \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(v)) + \sum_{i+j=n} f_i(\beta^{n-i}(v))d_j(\alpha^{n-j}(u))$ for all $u, v \in U$. Putting u by $-u$ in above expression and combining with it, we get

$$\eta_n(u)\alpha^n(v) + \mu(u,v)\alpha^n(u) = 0$$

for all $u, v \in U$. On right multiplying by $\eta_n(u)$ and using (2.11) and Lemma 2.1, we get

$$\eta_n(u)\alpha^n(v)\eta_n(u) = 0$$

for all $u, v \in U$. Since α^n is an automorphism, so we conclude that $\eta_n(u) = 0$ for all $u \in U$ i.e., $f_n(u^2) = \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(u))$ for all $u \in U$. Therefore in view of [5] Theorem 2.1, we get the required result. This completes the proof of the theorem. □

As special case of the above theorem, and which are of independent interests, are the following corollaries:

Corollary 2.3. *[14, Theorem 2.3] Let R be a 2-torsion free semiprime $*$-ring. Suppose there exists a family of additive mappings $F = (f_i)_{i\in\mathbb{N}_0}$ of R associated with some higher derivation $D = (d_i)_{i\in\mathbb{N}_0}$ of R such that $f_0 = id_R$, and the relation $f_n(xx^*) = \sum_{i+j=n} f_i(x)d_j(x^*)$ is fulfilled for each $n \in \mathbb{N}_0$ and for all $x \in R$. Then, F is a generalized higher derivation.*

Corollary 2.4. *Let R be a 2-torsion free semirprime ring with involution. Suppose there exists a family of additive mappings $D = (d_i)_{i\in N_0}$ of R, where α, β are commuting automorphisms on R, such that $d_n(xx^*) = \sum_{i+j=n} d_i(\alpha^{n-i}(x))d_j(\beta^{n-j}(x^*))$ is fulfilled for each $n \in N$ and for all $x \in R$. Then, d is a (α, β)-higher derivation on R.*

3 Conclusions

This paper deals with the following $*$-differntial identity $f_n(xx^*) = \sum_{i+j=n} f_i(\alpha^{n-i}(x))d_j(\beta^{n-j}(x^*))$ on a square close Lie ideal u of a prime ring R, where $F = (f_i)_{i\in N_0}$ is a family of additive maps associated with (α, β)-higher derivations of u into R. We inferred that F is a generalized (α, β)-higher derivation of u into R.

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Competing Interests

Author has declared that there is no competing interests exists.

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