Asian Research Journal of Mathematics

15(4): 1-14, 2019; Article no.ARJOM.52367
ISSN: 2456-477X

An Approximate Solution for a Nonlinear Duffing – Harmonic Oscillator

Van Hieu – Dang[1*]

[1]*Department of Mechanics, Thai Nguyen University of Technology, Thai Nguyen, Viet Nam.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2019/v15i430154
Editor(s):
(1) Dr. Xingting Wang, Department of Mathematics, Howard University, Washington DC, USA.
Reviewers:
(1) Chunhua, Alabama State University, USA.
(2) Mohamed A. El-Beltagy, Cairo University, Egypt.
(3) Adel H. Phillips, Ain-Shams University, Egypt.
Complete Peer review History: http://www.sdiarticle4.com/review-history/52367

Original Research Article

Received: 20 August 2019
Accepted: 28 October 2019
Published: 05 November 2019

Abstract

The equivalent linearization method introduced by Caughey is a powerful tool for analyzing random oscillations. The method is also easy to apply for deterministic oscillations. However, with strong nonlinear systems, the error of this method is usually quite large and even not acceptable. In conjunction with a weighted averaging, the equivalent linearization method has shown more accuracy than the classical one in which the conventional averaging value is used. Combining advantages of the classical equivalent linearization method and accuracy of the weighted averaging, the proposed method has shown that it is a useful tool for analyzing nonlinear oscillations including strong nonlinear systems. In this paper, the proposed method is applied to analyze a nonlinear Duffing – harmonic oscillator. The present results are compared with the results obtained by using other analytical methods, exact results and numerical results.

Keywords: Equivalent linearization method; weighted averaging; nonlinear oscillators.

1 Introduction

Oscillation is a common phenomenon in nature and engineering. In engineering, oscillations occur in many fields, such as oscillation of machinery in mechanics, oscillation of buildings or bridges in the field of

**Corresponding author: E-mail: hieudv@tnut.edu.vn;*

construction, oscillation of traffic vehicles, or oscillation of circuits, and so on. Each oscillation phenomenon is often described mathematically by nonlinear differential equations. It is difficult to find exact solutions of these equations. Numerical methods have given us very effective tools to describe these oscillations. However, the frequency-amplitude relationship, the most important feature of oscillation problems, is very difficult to find by using numerical methods. Thus, approximate analytical methods are needed to analyze response of nonlinear oscillations. Some approximate analytical methods have been introduced recently, such as the Parameter Perturbation method (PPM) [1], the Variational Iteration Method (VIM) [2], the Homotopy Perturbation Method (HPM) [3], the Energy Balance Method (EBM) [4], the Parameter Expansion Method (PEM) [5], the Variational Approach (VA) [6], the Hamiltonian Approach (HA) [7]; the Homotopy Analysis Method (HAM) [8], the Amplitude-Frequency Formulation (AFF) [9,10] and the Equivalent Linearization method (ELM) [11]. These approximate analytical methods are effectively analytical tools for investigating nonlinear oscillation problems.

The Equivalent Linearization method was proposed by Caughey [11] in 1963 for purpose of analyzing random nonlinear oscillations. This method is based on the classical ELM of Kryloff and Bogoliubov for deterministic nonlinear systems [12]. ELM [11] is used very commonly for analyzing random nonlinear systems. However, this method is only effective for weak nonlinear systems. There were some developments of this method such as the non-parametric ELM based on first-order reliability method [13], the tail-equivalent linearization method [14], ELM using Gaussian mixture [15] and the regulated Gaussian ELM [16, 17].

In a conventional approach of ELM, the nonlinear system is replaced by an equivalent linear system in which the parameters of the equivalent linear system are determined by minimizing a measure of the discrepancy between the responses of the nonlinear and linear systems called the mean-square criterion [11]. To improve the accuracy of ELM, some extensions of the mean-square criterion was proposed such as a dual criterion [18,19] and local mean square error criterion [20]. Recently, Anh et al. [21,22] introduced a new development of ELM based on the weighted averaging value. Instead of using the conventional/classical averaging value [12] in which the averaging value is calculated in one period of oscillation for harmonic functions, Anh et al. proposed a new method for estimating the averaging value of deterministic functions by introducing a weighted coefficient function, the new averaging value is called the weighted one. With this new proposal, accuracy of the ELM has been greatly improved. The new proposed method has been applied very effectively in analysis of deterministic strong nonlinear oscillations [22-25]. In this paper, the proposed method is further extended in analysis of a Duffing-harmonic oscillator. The present solutions are compared with the ones achieved by different approximate analytical methods, the numerical solutions and the exact solutions.

2 The Nonlinear Duffing-harmonic Oscillator

In this paper, we consider a nonlinear Duffing-harmonic oscillator given as [26]:

$$\ddot{u} + k_1 u + \frac{k_3 u^3}{1 + k_2 u^2} = 0, \tag{1}$$

with the initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0. \tag{2}$$

where k_1, k_2 and k_3 are constants.

The equivalent linearization method will be applied to find the appromimate analytical solution of Eq. (1). First, we rewrite Eq. (1) in the following form:

$$(1 + k_2 u^2)\ddot{u} + k_1 u + k_1 k_2 u^3 + k_3 u^3 = 0. \tag{3}$$

We introduce the linear form of the nonlinear Eq. (1) as follows:

$$\ddot{u} + \omega^2 u = 0. \tag{4}$$

where ω is known as the approximate frequency of oscillation and needs to be determined. The coefficient ω can be found by many different criteria, in which the most common criterion is the mean square criterion requiring the mean square of equation error between the nonlinear equation (3) and the linear equation (4) be minimum [11]:

$$\left\langle e^2(u) \right\rangle = \left\langle \left(k_2 u^2 \ddot{u} + k_1 u + (k_1 k_2 + k_3) u^3 - \omega^2 u \right)^2 \right\rangle \to \min_{\omega^2}. \tag{5}$$

where $\langle \square \rangle$ is the averaging operator; the mean square criterion (5) states that when the error between the nonlinear equation (3) and the linear one (4) is minimum, the solution of the linear equation will be closest to the solution of the nonlinear equation. From the condition $\partial \left\langle e^2(u) \right\rangle / \partial \omega^2$, we get a equation for determining the coefficient ω^2 as follows:

$$k_2 \left\langle u^3 \ddot{u} \right\rangle + k_1 \left\langle u^2 \right\rangle + (k_1 k_2 + k_3) \left\langle u^4 \right\rangle - \omega^2 \left\langle u^2 \right\rangle = 0. \tag{6}$$

The periodic solution of linear Eq. (4) with the initial conditions (2) has the form:

$$u = A\cos(\omega t). \tag{7}$$

The classical averaging value of a harmonic function can be get as [12]:

$$\left\langle f(u(t)) \right\rangle_c = \left\langle f(A\cos(\omega t)) \right\rangle_c = \frac{1}{T}\int_0^T f\left(A\cos(\omega t)\right) dt = \frac{1}{2\pi}\int_0^{2\pi} f\left(A\cos(\tau)\right) d\tau \tag{8}$$

here $T = 2\pi / \omega$ is the period of oscillation and $\tau = \omega t$. The averaging values in Eq. (8) is called the classical averaging value which often leads to solutions of strong nonlinear systems with large and sometimes unacceptable errors. In order to improve this issue, in this paper, a new method is proposed to determine the averaging values in Eq. (6) called weighted averaging value. As proposed by Anh et al. [21, 22], the averaging value can be determined by:

$$\left\langle f(t) \right\rangle_w = \int_0^d h(t) f\left(t\right) dt, \tag{9}$$

where $h(t)$ is the weighted coefficient function satisfying the following condition [21, 22]:

$$\int_0^d h(t) dt = 1. \tag{10}$$

A specific weighted coefficient function will be used in this paper as follows:

$$h(t) = s^2 \omega^2 t e^{-s\omega t}, \tag{11}$$

herein, *s* is a positive constant. We can easily check that $h(t)$ given in Eq. (11) satisfies the condition (10) as $d \to \infty$. Based on the weighted coefficient function $h(t)$ given in Eq. (11), the weighted averaging value of a function $f(\omega t)$ can be calculated by using Eq. (9):

$$\left\langle f(\omega t)\right\rangle_w = \int_0^{+\infty} s^2\omega^2 t e^{-s\omega t} f\left(\omega t\right)dt = \int_0^{+\infty} s^2\tau e^{-s\tau} f\left(\tau\right)d\tau, \tag{12}$$

We can see that the weighted averaging value of the function *f(ωt)* is the Laplace transform of function $s^2 tf(t)$ as follows:

$$\left\langle f(\omega t)\right\rangle_w = \int_0^{+\infty} s^2\tau e^{-s\tau} f\left(\tau\right)d\tau = F(s). \tag{13}$$

The averaging value of function *f(ωt)* does not depend on the time *t* but depends on the parameter *s* (in expression of the weighted coefficient function *h(t)*). The parameter *s* is called the adjustment parameter, accuracy of the obtained solution depends on the choice of value of the parameter *s*. The choice of value of the parameter *s* has been examined by authors in many cases [22-25], the obtained results will be very accurate when *s* = 2.

Based on the periodic solution in Eq. (7), we can calculate averaging operators in Eq. (6) by using Eqs. (12) and (13):

$$\begin{aligned}\left\langle u^2\right\rangle_w &= \left\langle A^2\cos^2(\omega t)\right\rangle_w = \int_0^{+\infty} A^2 s^2\omega^2 t e^{-s\omega t}\cos^2(\omega t)dt = \int_0^{+\infty} A^2 s^2\tau e^{-s\omega\tau}\cos^2(\tau)d\tau \\ &= A^2\frac{s^4+2s^2+8}{(s^2+4)^2},\end{aligned} \tag{14}$$

$$\begin{aligned}\left\langle u^4\right\rangle_w &= \left\langle A^4\cos^4(\omega t)\right\rangle_w = \int_0^{+\infty} A^4 s^2\omega^2 t e^{-s\omega t}\cos^4(\omega t)dt = \int_0^{+\infty} A^4 s^2\tau e^{-s\omega\tau}\cos^4(\tau)d\tau \\ &= A^4\frac{s^8+28s^6+248s^4+416s^2+1536}{(s^2+4)^2(s^2+16)^2},\end{aligned} \tag{15}$$

$$\begin{aligned}\left\langle u^3\ddot{u}\right\rangle &= -\left\langle A^4\omega^2\cos^4(\omega t)\right\rangle = -\int_0^{+\infty} A^4\omega^2 s^2\omega^2 t e^{-s\omega t}\cos^4(\omega t)dt \\ &= -\int_0^{+\infty} A^4\omega^2 s^2\tau e^{-s\omega\tau}\cos^4(\tau)d\tau = -A^4\omega^2\frac{s^8+28s^6+248s^4+416s^2+1536}{(s^2+4)^2(s^2+16)^2}.\end{aligned} \tag{16}$$

Substituting Eqs. (14)-(16) into Eq. (6), we obtain the approximate frequency of oscillation:

$$\omega = \sqrt{\frac{k_1(s^4+2s^2+8)(s^2+16)^2+(k_1k_2+k_3)(s^8+28s^6+248s^4+416s^2+1536)A^2}{(s^4+2s^2+8)(s^2+16)^2+k_2(s^8+28s^6+248s^4+416s^2+1536)A^2}}. \tag{17}$$

When s=2, we have the approximate frequency:

$$\omega = \sqrt{\frac{k_1 + 0.72(k_1k_2 + k_3)A^2}{1 + 0.72k_2A^2}}. \tag{18}$$

Thus, from Eq. (7), the approximate solution of oscillation can be get as follows:

$$u(t) = A\cos\left(\sqrt{\frac{k_1 + 0.72(k_1k_2 + k_3)A^2}{1 + 0.72k_2A^2}}\ t\right). \tag{19}$$

To show accuracy of the present results, we compare the present solution given in Eq. (19) with the numerical one using the 4th–order Runge-Kutta method, the result is presented in Fig. 1 for k_1=20, k_2=100 and k_3=20.

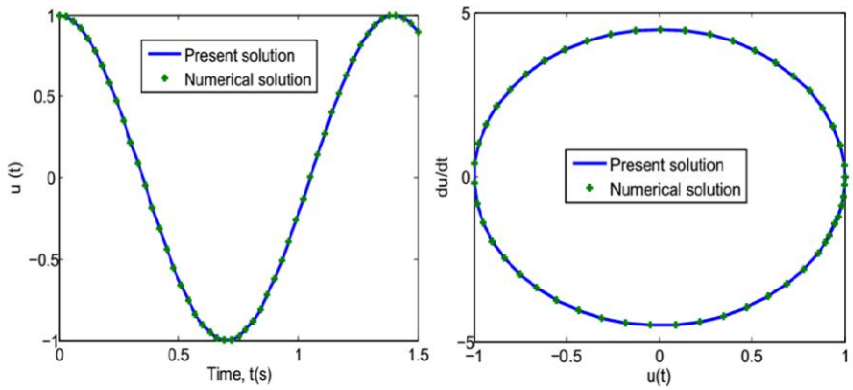

Fig. 1. Comparison between the present solution with the numerical solution

For purpose of comparing the current solution with published solutions using different approximate analytical methods, we consider some following cases:

2.1 Case 1: k_2=0

If k_2=0, from Eq. (1) we have a cubic Duffing oscillator:

$$\ddot{u} + k_1u + k_3u^3 = 0. \tag{20}$$

Eq. (20) was analyzed by authors in ref. [24]. From Eqs. (18) and (19), we can get the approximate frequency:

$$\omega = \sqrt{k_1 + 0.72k_3A^2}, \tag{21}$$

and the approximate solution of oscillation as follows:

$$u(t) = A\cos\left(\sqrt{k_1 + 0.72k_3A^2}\ t\right). \tag{22}$$

The approximate frequency of Eq. (20) can be achieved by using the Parametrized Perturbation method (PPM) [27] and the Energy Balance method (EBM) [28] as follows:

$$\omega_{PPM} = \omega_{EBM} = \sqrt{k_1 + 0.75k_3A^2}. \tag{23}$$

The exact frequency of this oscillation is given by [27]:

$$\omega_{exact} = \frac{2\pi}{4\sqrt{2}\int_0^{\pi/2} \frac{dt}{\sqrt{k_3A^2\left(1+\cos^2(t)\right)+2k_1}}}. \tag{24}$$

Accuracy of the present solution is shown in Table 1 and Fig. 2. For some values of the initial amplitude (A) and the coefficients of system (k_1 and k_3), values of approximate frequencies and exact frequency are showed in Table 1. We can see that the relative errors of the EBM and PPM frequencies reach to 2.2% while the relative error of the current frequency is only 0.15% when the initial amplitude of oscillation increases. With the fixed coefficients k_1=20, k_3=20 and the initial amplitude A=5, time history and phase portrait of approximate solutions and exact solution of the cubic Duffing oscillation are presented in Fig. 2.

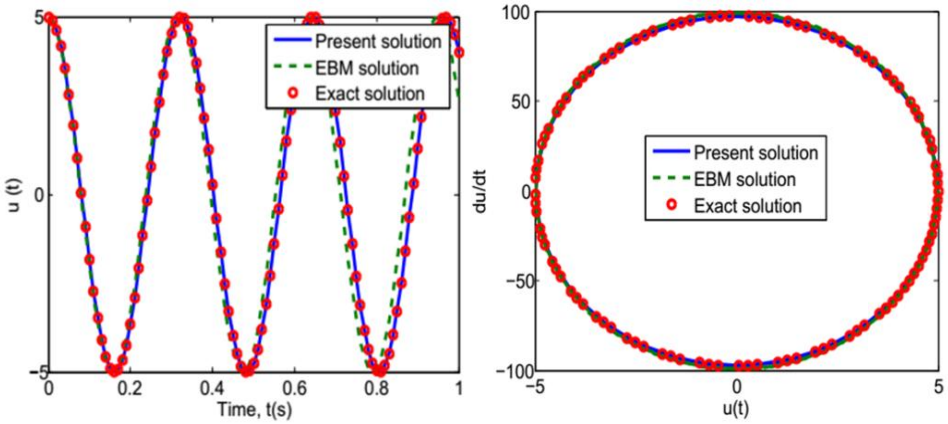

Fig. 2. Time history and phase portrait of the cubic Duffing oscillation

Table 1. Comparison of the approximate frequencies with the exact frequency, case 1

A	k_1	k_3	ω_{exact} [27]	ω_{EBM}, ω_{PPM} [27, 28]	R. error (%)	$\omega_{present}$	R. error (%)
0.1	1	1	1.0037	1.0037	0.0001	1.0035	0.0147
	10	10	3.1741	3.1741	0.0001	3.1736	0.0147
	100	100	10.0374	10.0374	0.0001	10.0359	0.0147
1	1	1	1.3177	1.3228	0.3869	1.3114	0.4771
	10	10	4.1671	4.1833	0.3869	4.1472	0.4771
	100	100	13.1777	13.2287	0.3869	13.1148	0.4771
10	1	1	8.5335	8.7177	2.1586	8.5440	0.1220
	10	10	26.9855	27.5680	2.1586	27.0185	0.1220
	100	100	85.3358	87.1779	2.1586	85.4400	0.1220
50	1	1	42.3729	43.3128	2.2179	42.4381	0.1538
	10	10	133.9951	136.9671	2.21796	134.2013	0.1538
	100	100	423.7299	433.1281	2.2179	424.3819	0.1538

2.2 Case 2: k_1=0

From Eq. (1), we have a Duffing oscillator with a rational elastic term:

$$\ddot{u} + \frac{k_3 u^3}{1 + k_2 u^2} = 0. \tag{25}$$

From Eq. (18), the approximate frequency can be get as:

$$\omega = \sqrt{\frac{0.72 k_3 A^2}{1 + 0.72 k_2 A^2}}. \tag{26}$$

and from Eq. (19), we can obtain the approximate solution of oscillation:

$$u(t) = A\cos\left(\sqrt{\frac{0.72 k_3 A^2}{1 + 0.72 k_2 A^2}}\, t\right). \tag{27}$$

With k_2=1 and k_3=1, the approximate frequency of oscillation given in Eq. (25) was achieved by using the Parameter Expansion method (PEM) [27], and the exact frequency of oscillation was given in ref. [29].

Table 2 shows comparison of the present frequency and the PEM frequency with the exact frequency for some values of the initial amplitude (A). From Table 2, we can see accuracy of the present solution. Responses of the oscillation obtained by using different analytical methods and the numerical method are showed in Fig. 3 with k_1=20 and k_3=20.

Table 2. Comparison of the approximate frequencies with the exact frequency, case 2

A	ω_{exact} [29]	ω_{PEM} [27]	R. error (%)	$\omega_{present}$	R. error (%)
0.01	0.0085	0.0087	2.2432	0.0085	0.0000
0.05	0.0423	0.0432	2.2212	0.0424	0.1654
0.1	0.0844	0.0863	2.2396	0.0846	0.1896
0.5	0.3874	0.3974	2.5789	0.3906	0.8826
1	0.6368	0.6547	2.8063	0.6469	1.6034
5	0.9669	0.9744	0.7622	0.9733	0.6567
10	0.9909	0.9934	0.2493	0.9931	0.2230

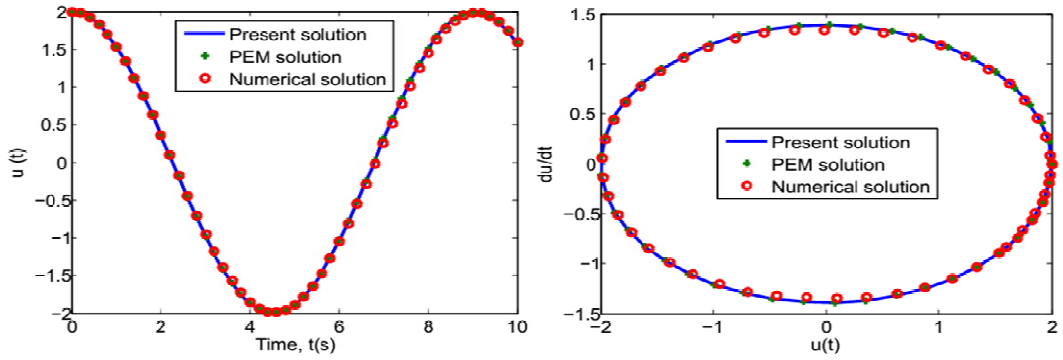

Fig. 3.Time history and phase portrait of the Duffing oscillation with rational elastic term

2.3 Case 3: k_1=-1, k_2=0 and k_3=1

We have a Duffing oscillator with double-well potential (negative linear stiffness):

$$\ddot{u} - u + u^3 = 0. \tag{28}$$

System (28) has three equilibrium points: the central equilibrium point $u = 0$ is unstable and the other two equilibrium points $u = \pm 1$ are stable. The periodic solutions of this oscillation depend on the initial amplitude A. For the case of $0 < A < 1$ and $1 < A < \sqrt{2}$, the equilibrium points $u = \pm 1$ are stable. And, for the case of $A > \sqrt{2}$, the periodic solution is symmetric and extends across three equilibrium points.

For the case of $A > \sqrt{2}$, from Eq. (18), the approximate frequency of oscillation can be obtained as:

$$\omega = \sqrt{-1 + 0.72 k_3 A^2}. \tag{29}$$

The present period $T_{present}$, the approximate period achieved by the Energy Balance method T_{EBM} [26] and the exact period T_{exact} [30] are listed in Table 3 for some values of the initial amplitude (A). The exact solution [30], the approximate solution achieved by the Energy Balance method [26] and the present solution are plotted in Fig.4 for the initial amplitude A=1.5. Again we can see accuracy of the present solution.

Table 3. Comparison of the approximate periods with the exact period for double-well Duffing oscillation ($A > \sqrt{2}$)

A	T_{exact} [30]	T_{EBM} [26]	R. error (%)	$T_{present}$	R. error (%)
1.42	15.0844	8.7784	41.8047	9.3477	38.0306
1.45	11.2132	8.2725	26.2253	8.7656	21.8278
1.5	9.2237	7.5778	17.8442	7.9797	13.4869
1.7	6.3528	5.8150	8.4655	6.0438	4.8639
2	4.6857	4.4429	5.1817	4.5825	2.2024
5	1.5286	1.4914	2.4335	1.5239	0.3074
10	0.7471	0.7304	2.2353	0.7457	0.1873
50	0.1484	0.1451	2.2237	0.1481	0.2021
100	0.0742	0.0726	2.1563	0.0741	0.1347
100	0.0074	0.0073	1.3513	0.0074	0.0000

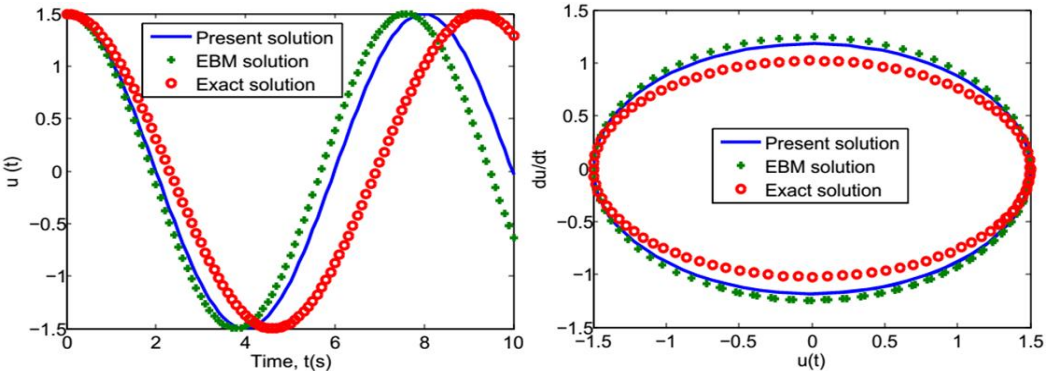

Fig. 4.Time history and phase portrait of responds of the Duffing oscillation with doule-well potential, A=1.5

- For the case of $1 < A < \sqrt{2}$, we introduce a new variable:

$$x = u - 1. \tag{30}$$

Substituting Eq. (30) into Eq. (28), we get:

$$\ddot{x} + 2x + 3x^2 + x^3 = 0, \tag{31}$$

with the initial conditions:

$$x(0) = \widehat{A},\ \dot{x}(0) = 0, \tag{32}$$

where $\widehat{A} = A - 1$. We will find the approximate solution of Eq. (31), the linear equation of Eq. (31) has the form:

$$\ddot{x} + \omega^2 x = 0. \tag{33}$$

Employing the mean square criterion, the coefficient ω^2 can be get as:

$$\omega^2 = \frac{2\left\langle x^2 \right\rangle + 3\left\langle x^3 \right\rangle + \left\langle x^4 \right\rangle}{\left\langle x^2 \right\rangle}. \tag{34}$$

Using the weighted averaging value, averaging operators in Eq. (34) can be obtained as:

$$\left\langle x^2 \right\rangle_w = \widehat{A}^2 \frac{s^4 + 2s^2 + 8}{(s^2 + 4)^2}. \tag{35}$$

$$\left\langle x^3 \right\rangle_w = \widehat{A}^3 \frac{s^2(s^6 + 11s^4 + 43s^2 - 63)}{(s^2 + 1)^2 (s^2 + 9)^2}. \tag{36}$$

$$\left\langle x^4 \right\rangle_w = \widehat{A}^4 \frac{s^8 + 28s^6 + 248s^4 + 416s^2 + 1536}{(s^2 + 4)^2 (s^2 + 16)^2}. \tag{37}$$

Substituting Eqs. (35)-(37) into Eq. (34) and with *s*=2, we get the approximate frequency of oscillation for this case:

$$\omega = \sqrt{2 + 1.9824\widehat{A} + 0.72\widehat{A}^2}. \tag{38}$$

Note that $\widehat{A} = A - 1$, from Eq. (38), the approximate frequency is given as:

$$\omega = \sqrt{2 + 1.9824(A - 1) + 0.72(A - 1)^2}. \tag{39}$$

Thus, the approximate solution for this case can be get:

$$u(t) = (A - 1)\cos\left(\sqrt{2 + 1.9824(A - 1) + 0.72(A - 1)^2}\, t\right) + 1. \tag{40}$$

To show accuracy of the obtained solution, the exact period T_{exact} [30], the approximate period obtained by the Energy Balance method T_{EBM} [26] and the present period $T_{present}$ are showed in Table 4 for some values of the initial amplitude A. We can conclude that the proposed method gives more excellent approximate periods than the EBM for the oscillation amplitude $1 < A < \sqrt{2}$.

Table 4. Comparison of the approximate periods with the exact period for double-well Duffing oscillation ($1 < A < \sqrt{2}$)

A	T_{exact} [30]	T_{EBM} [26]	R. error (%)	$T_{present}$	R. error (%)
1.05	4.3061	4.3045	0.0373	4.3349	0.0067
1.1	4.1781	4.1748	0.0781	4.2309	0.0126
1.15	4.0582	4.0530	0.1267	4.1309	0.0179
1.2	3.9460	3.9384	0.1923	4.0347	0.0225
1.25	3.8417	3.8303	0.2961	3.9420	0.0261
1.3	3.7468	3.7282	0.4964	3.8529	0.0283
1.35	3.6688	3.6316	1.0139	3.7671	0.0268
1.4	3.6897	3.5399	4.0576	3.6845	0.0014
1.41	3.8506	3.5222	8.5261	3.6684	0.0473
1.412	3.9755	3.5164	11.548	3.6652	0.0781

Fig. 5 shows comparison of the present solution and the EBM solution with the exact solution for the initial amplitude A=1.4.

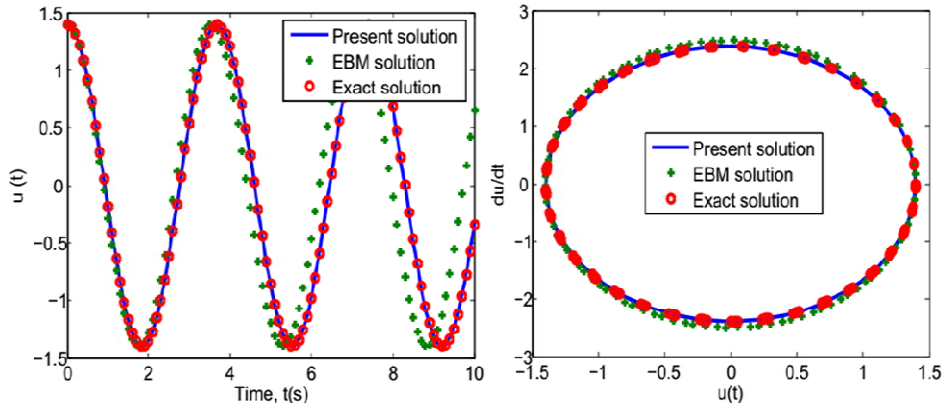

Fig. 5. Time history and phase portrait of responds of the Duffing oscillation with doule-well potential, A=1.4

2.4 Case 4: k_2=1

With k_2=1, Eq. (1) becomes:

$$\ddot{u} + k_1 u + \frac{k_3 u^3}{1+u^2} = 0. \tag{41}$$

From Eq. (18), the approximate frequency of oscillation given by Eq. (41) can be obtained as:

$$\omega = \sqrt{\frac{k_1 + 0.72(k_1 + k_3)A^2}{1 + 0.72A^2}}. \tag{42}$$

The approximate frequency of oscillation for this case can be achieved by using the Energy Balance method [26] as follows:

$$\omega_{EBM} = \sqrt{(k_1 + k_3) + 2\frac{k_3}{A^2}\left[\ln\left(1 + A^2/2\right) - \ln(1 + A^2)\right]}. \tag{43}$$

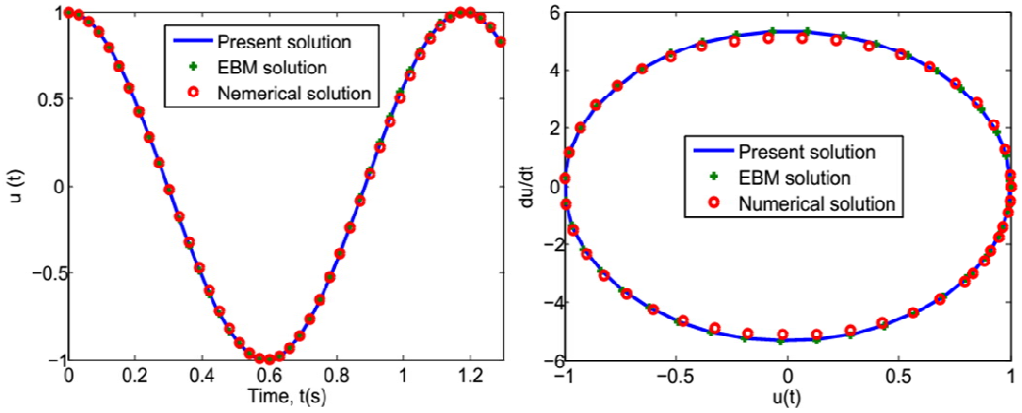

Fig. 6. Time history and phase portrait of Duffing-harmonic oscillation with A=1

Comparisons of the present solution and EBM solution for this oscillation are presented in Table 5 and Fig. 6. We can see very good agreement of two approximate solutions. Fig. 6 is plotted with k_1=20 and k_3=20.

Table 5. Values of the approximate frequencies achieved by the Energy Balance method and the current method for Duffing-harmonic oscillation

k_1	k_3	A	ω_{EBM} [26]	$\omega_{present}$
1	1	0.1	1.0037	1.0037
		1	1.1936	1.1910
		5	1.3956	1.3955
		10	1.4094	1.4094
		50	1.4140	1.4140
		100	1.4142	1.4142
1	100	0.1	1.3207	1.3095
		1	6.5927	6.5468
		5	9.7855	9.7845
		10	9.9817	9.9815
		50	10.047	10.0471
		100	10.0492	10.0492
100	1	0.1	10.0004	10.0004
		1	10.0212	10.0209
		5	10.0473	10.0473
		10	10.0492	10.0492
		50	10.0498	10.0498
		100	10.0499	10.0499
100	100	0.1	10.0371	10.0357
		1	11.9358	11.9105
		5	13.9555	13.9548
		10	14.0937	14.0936
		50	14.1402	14.1402
		100	14.1416	14.1416

4 Conclusion

In this work, the equivalent linearization method in conjunction with a weighted averaging is applied in analysis of a nonlinear Duffing – harmonic oscillator. This method is a development of the classical equivalent linearisation method. In the proposed method, the averaging value is calculated in a new way called the weighted averaging value by introducing a weighted coefficient function $h(t)$. Accuracy of the proposed method is verified by comparing the obtained results with the published ones for some specific cases. The proposed method is very simple for applying and gives the results in very high accuracy for both nonlinear systems. The present method will be an effective tool in analyzing nonlinear oscillations.

Acknowledgements

This work is supported by Thai Nguyen University of Technology (TNUT) (no. "**T2018-B27**").

Competing Interests

Author has declared that no competing interests exist.

References

[1] He JH. Some new approaches to duffing equation with strongly and high order nonlinearity (ii) parametrized perturbation technique. Communications in Nonlinear Science and Numerical Simulation. 1999;4(1):81–83.

[2] He JH. Variational iteration method-a kind of non-linear analytical technique: some examples. International Journal of Non-Linear Mechanics. 1999;34(4):699–708.

[3] He JH. Homotopy perturbation technique. Computer methods in applied mechanics and engineering. 1999;178(3-4):257–262.

[4] He JH. Preliminary report on the energy balance for nonlinear oscillations. Mechanics Research Communications. 2002;29(2-3):107–111.

[5] He JH. An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering. International Journal of Modern Physics B (IJMPB). 2008;22(21):3487–3578.

[6] He JH. Variational approach for nonlinear oscillators. Chaos, Solitons & Fractals. 2007;34(5):1430–1439.

[7] He JH. Hamiltonian approach to nonlinear oscillators. Physics Letters A. 2010;374(23):2312–2314.

[8] Liao S. Homotopy analysis method and its application [thesis]. Shanghai Jiao Tong University, Shanghai, China; 1992.

[9] He JH. Nonperturbative methods for strongly nonlinear problems. Dissertation.de-Verlag im Internet GmbH. 2006;290.

[10] He JH. Amplitude-Frequency Relationship for Conservative Nonlinear Oscillators with Odd Nonlinearities. International Journal of Applied and Computational Mathematics. 2017;3:1557-1560.

[11] Caughey TK. Equivalent linearization technique. The Journal of the Acoustical Society of America. 1963;35:1706–1711.

[12] Krylov N, Bogoliubov N. Introduction to nonlinear mechanics. New York: Princenton University Press; 1943.

[13] Der Kiureghian A. First- and second-order reliability methods. In: Nikolaidis E, Ghiocel DM, Singhal S, editors. Engineering design reliability handbook. Boca Raton (FL): CRC Press. 2005;Chapter 14.

[14] Fujimura K, Der Kiureghian A. Tail-equivalent linearization method for nonlinear random vibration. Probabilistic Engineering Mechanics. 2007;22:63–76.

[15] Ziqi Wang, Junho Song. Equivalent linearization method using Gaussian mixture (GM-ELM) for nonlinear random vibration analysis. Structural Safety; 2016.

[16] Anh ND, Di Paola M. Some Extensions of Gaussian Equivalent Linearization. In: International Confernce on Nonlinear Stochastic Dynamics; Hanoi, Vietnam, 7–10 December. 1995;5–16.

[17] Isaac Elishakoff, Lova Andriamasy, Melanie Dolley. Application and extension of the stochastic linearization by Anh and Di Paola. Acta Mechanica. 2009;204(1-2):89–98.

[18] Anh ND. Duality in the analysis of responses to nonlinear systems. Vietnam Journal of Mechanics. 2010;32:263–266.

[19] Anh ND, Hieu NN, Linh NN. A dual criterion of equivalent linearization method for nonlinear systems subjected to random excitation. Acta Mechanica. 2012;223(3):645–654.

[20] Anh ND, Hung LX, Viet LD. Dual approach to local mean square error criterion for stochastic equivalent linearization. Acta Mechanica. 2013;224(2):241–253.

[21] Anh ND. Short Communication - Dual approach to averaged values of functions: a form for weighting coefficient. Vietnam Journal of Mechanics. 2015;37(2):145 –150.

[22] Anh ND, Hai NQ, Hieu DV. The Equivalent Linearization Method with a Weighted Averaging for Analyzing of Nonlinear Vibrating Systems. Latin American Journal of Solids and Structures. 2017; 14:1723- 1740.

[23] Dang Van Hieu, Ninh Quang Hai and Duong The Hung. Analytical Approximate Solutions For Oscillators With Fractional Order Restoring Force And Relativistic Oscillators. International Journal of Innovative Science, Engineering & Technology. 2017;4(12):28-35.

[24] Hieu DV, Hai NQ, Hung DT. The Equivalent Linearization Method with a Weighted Averaging for Solving Undamped Nonlinear Oscillators. Journal of Applied Mathematics; 2018.

[25] Hieu DV, Hai NQ. Analyzing of Nonlinear Generalized Duffing Oscillators Using the Equivalent Linearization Method with a Weighted Averaging. Asian Research Journal of Mathematics. 2018; 9(1):1-14.

[26] Momeni M, Jamshidi N, Barari A, Ganji DD. Application of He's energy balance method to Duffing-harmonic oscillators. International Journal of Computer Mathematics. 2010;88(1):135-144.

[27] Bayat M, Pakar I, Domairry G. Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review. Latin American Journal of Solids and Structures. 2012;9(2):145–234.

[28] Younesian D, Askari H, Saadatnia Z, KalamiYazdi M. Frequency analysis of strongly nonlinear generalized Dufing oscillators using He's frequency-amplitude formulation and He's energy balance method. Computers & Mathematics with Applications: An International Journal. 2010;59(9):3222–3228.

[29] Ganji SS, Ganji DD, Davodi AG, Karimpour S. Analytical solution to nonlinear oscillation system of the motion of a rigid rod rocking back using max–min approach. Applied Mathematical Modelling. 2010;34:2676–2684.

[30] Wu BS, Sun WP, Lim CW. Analytical approximations to the double-well Duffing oscillator in large amplitude oscillations. Journal of Sound and Vibration. 2007;307:953–960.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/52367