Asian Research Journal of Mathematics

15(4): 1-13, 2019; Article no.ARJOM.51185

ISSN: 2456-477X

A New and Simple Condensation Method of Computing Determinants of Large Matrices and Solving Large Linear Systems

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Authors contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2019/v15i430159

Editor(s):

(1) Dr. Danilo Costarelli, Department of Mathematics and Computer Science, University of Perugia, Italy.

Reviewers:

(1) A. Ayeshamariam, Khadir Mohideen College, India.

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Complete Peer review History: http://www.sdiarticle4.com/review-history/51185

Received: 26 June 2019

Accepted: 04 September 2019

Published: 11 November 2019

Original Research Article

Abstract

The chief object of this work present a new and simple condensation method of finding determinants of large matrices and solving large linear systems.

Keywords: Condensation; matrices; determinants; linear systems.

2010 Mathematics Subject Classification: 03E10; 03H05; 03H10.

1 Introduction

There does seem to be currently available some condensation methods of evaluating large determinants. Two such methods are the famous Dodgson's condensation and the reknown Chió's condensation [1], [2], [3]. Both methods in the case of an $n \times n$ matrix are to construct an $(n-1) \times (n-1)$ matrix,

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an $(n-2)\times(n-2)$, and so on, finishing with a 1×1 matrix, which has one entry, the determinant of the original matrix.

The aim of this paper, however, is to develop a similar but simpler condensation technique of evaluating determinants of large matrices and solving large linear systems. This method is novel and has the quality of being brief and straightforward.

The remainder of this paper consists of two sections. The first section discusses how the method is developed and used to evaluate the determinant of nth order. The second section deals with the use of the same technique in obtaining the solution sets of systems of linear equations. In both sections ample instances are given to illustrate the utility of the proposed technique.

2 New Condensation Method of Evaluating Determinant

In this section we develop a new condensation method of evaluating determinants of large matrices. Consider the nth order determinant [4], [5], [6].

$$\left|\mathbf{A_n}\right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

Subtract $\frac{a_{11}}{a_{21}}$ times the 2nd row from the 1st row. Subtract $\frac{a_{31}}{a_{21}}$ times the 2nd row from 3rd rows. Subtract $\frac{a_{41}}{a_{31}}$ times the 3rd row from the 4th row. Subtract $\frac{a_{51}}{a_{41}}$ times the 4th row from the 5th row. The procedure continues until the last stage where we subtract $\frac{a_{n1}}{a_{(n-1)1}}$ times the $(n-1)$th row from the nth row. So, we get

$$\left|\mathbf{A_n}\right| = -a_{21}\begin{vmatrix} a_{12}-\dfrac{a_{11}}{a_{21}}a_{22} & a_{13}-\dfrac{a_{11}}{a_{21}}a_{23} & \dots & a_{1n}-\dfrac{a_{11}}{a_{21}}a_{2n} \\ a_{32}-\dfrac{a_{31}}{a_{21}}a_{22} & a_{33}-\dfrac{a_{31}}{a_{21}}a_{23} & \dots & a_{3n}-\dfrac{a_{31}}{a_{21}}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2}-\dfrac{a_{n1}}{a_{(n-1)1}}a_{(n-1)2} & a_{n3}-\dfrac{a_{n1}}{a_{(n-1)1}}a_{(n-1)3} & \ddots & a_{nn}-\dfrac{a_{n1}}{a_{(n-1)1}}a_{(n-1)n} \end{vmatrix}$$

$$= -a_{21}\times$$

$$\begin{vmatrix} \dfrac{a_{12}a_{21}-a_{11}a_{22}}{a_{21}} & \dfrac{a_{13}a_{21}-a_{11}a_{23}}{a_{21}} & \dots & \dfrac{a_{1n}a_{21}-a_{11}a_{2n}}{a_{21}} \\ \dfrac{a_{21}a_{32}-a_{23}a_{31}}{a_{21}} & \dfrac{a_{21}a_{33}-a_{23}a_{31}}{a_{21}} & \dots & \dfrac{a_{21}a_{3n}-a_{2n}a_{31}}{a_{21}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{a_{(n-1)1}a_{n2}-a_{(n-1)2}a_{n1}}{a_{(n-1)1}} & \dfrac{a_{(n-11)}a_{n3}-a_{(n-1)3}a_{n1}}{a_{(n-1)1}} & \dots & \dfrac{a_{(n-1)1}a_{nn}-a_{(n-1)n}a_{n1}}{a_{(n-1)1}} \end{vmatrix}$$

$$= -a_{21} \times \begin{vmatrix} -\frac{1}{a_{21}}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & -\frac{1}{a_{21}}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & -\frac{1}{a_{21}}\begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \frac{1}{a_{21}}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \frac{1}{a_{21}}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \frac{1}{a_{21}}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{(n-1)1}}\begin{vmatrix} a_{(n-1)1} & a_{(n-1)2} \\ a_{n1} & a_{n2} \end{vmatrix} & \frac{1}{a_{(n-1)1}}\begin{vmatrix} a_{(n-1)1} & a_{(n-1)3} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \frac{1}{a_{(n-1)1}}\begin{vmatrix} a_{(n-1)1} & a_{(n-1)n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

$$= \frac{1}{a_{21} \cdot a_{31} \cdot \cdots \cdot a_{(n-1)1}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{(n-1)1} & a_{(n-1)2} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{(n-1)1} & a_{(n-1)3} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{(n-1)1} & a_{(n-1)n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

Therefore,

$$|\mathbf{A_n}| = \frac{1}{a_{21} \cdot a_{31} \cdot \cdots \cdot a_{(n-1)1}} |\mathbf{A_{n-1}}|.$$

The determinant $|\mathbf{A_n}|$ is thus,

$$|\mathbf{A_n}| = \frac{1}{f_1} |\mathbf{A_{n-1}}|$$

where $f_1 = a_{21} \cdot a_{31} \cdot \ldots \cdot a_{(n-1)1}$.
Repeated application of the condensation method results in a second order determinant $|\mathbf{A_2}|$ which can be evaluated. The determinant $|\mathbf{A_n}|$ is then computed from the second order determinant.

The condensation process cannot be continued at any condensation stage where zeros appear in column 1 between the top-most element and bottom-most element, because infinite values would arise by applying the zeros as divisors. Since it is impossible for a determinant, whose elements are all real numbers , to be infinite,we must transform the determinant of that condensation stage such that no zero appears in column 1 between the top-most element and bottom-most element. To transform any determinant such that its absolute value remains the same, we apply any of the following properties of a determinant [4], [7]:

1. The value of the determinant remains unchanged, if the rows are interchanged into columns or the columns into rows.
2. If two rows or columns of the determinant are interchanged, the sign of the value of the determinant changes.
3. The value of the determinant is not changed by adding to each element of a row (or column) the corresponding element of another row (or column) multiplied by a constant factor.

The application of the properties of determinants to any condensation stage will never affect the final result.

Let us now demonstrate with examples how the new condensation method is used to hand-evaluate determinants. Suppose we wish to hand-evaluate the 3rd order determinant

$$\begin{vmatrix} 1 & 2 & 4 \\ 4 & 3 & 5 \\ 4 & 2 & 7 \end{vmatrix}.$$

We perform this as follows:

$$\begin{vmatrix}1 & 2 & 4\\4 & 3 & 5\\4 & 2 & 7\end{vmatrix} = \frac{1}{4}\begin{vmatrix}\begin{vmatrix}1 & 2\\4 & 3\end{vmatrix} & \begin{vmatrix}1 & 4\\4 & 5\end{vmatrix}\\ \begin{vmatrix}4 & 3\\4 & 2\end{vmatrix} & \begin{vmatrix}4 & 5\\4 & 7\end{vmatrix}\end{vmatrix} = \frac{1}{4}\begin{vmatrix}-5 & -11\\-4 & 8\end{vmatrix} = \frac{1}{4}\left|-84\right| = -21.$$

Let us now take an instance in which 0 appears as a divisor, namely

$$\begin{vmatrix}5 & 3 & 7 & 8\\0 & 4 & 1 & 2\\3 & 1 & 2 & 3\\1 & 2 & 5 & 5\end{vmatrix}.$$

Since $f_1 = 0 \times 3 = 0$, we interchange columns 1 and 2 elements Thus, we compute

$$\begin{vmatrix}5 & 3 & 7 & 8\\0 & 4 & 1 & 2\\3 & 1 & 2 & 3\\1 & 2 & 5 & 5\end{vmatrix} = -\begin{vmatrix}3 & 5 & 7 & 8\\4 & 0 & 1 & 2\\1 & 3 & 2 & 3\\2 & 1 & 5 & 5\end{vmatrix} = -\frac{1}{4\times 1}\begin{vmatrix}\begin{vmatrix}3 & 5\\4 & 0\end{vmatrix} & \begin{vmatrix}3 & 7\\4 & 1\end{vmatrix} & \begin{vmatrix}3 & 8\\4 & 2\end{vmatrix}\\ \begin{vmatrix}4 & 0\\1 & 3\end{vmatrix} & \begin{vmatrix}4 & 1\\1 & 2\end{vmatrix} & \begin{vmatrix}4 & 2\\1 & 3\end{vmatrix}\\ \begin{vmatrix}1 & 3\\2 & 1\end{vmatrix} & \begin{vmatrix}1 & 2\\2 & 5\end{vmatrix} & \begin{vmatrix}1 & 3\\2 & 5\end{vmatrix}\end{vmatrix} = -\frac{1}{4}\begin{vmatrix}-20 & -25 & -26\\12 & 7 & 10\\-5 & 1 & -1\end{vmatrix}$$

which becomes

$$\begin{vmatrix}5 & 3 & 7 & 8\\0 & 4 & 1 & 2\\3 & 1 & 2 & 3\\1 & 2 & 5 & 5\end{vmatrix} = -\frac{1}{4}\cdot\frac{1}{12}\begin{vmatrix}\begin{vmatrix}-20 & -25\\12 & 7\end{vmatrix} & \begin{vmatrix}-20 & -26\\12 & 10\end{vmatrix}\\ \begin{vmatrix}12 & 7\\-5 & 1\end{vmatrix} & \begin{vmatrix}12 & 10\\-5 & -1\end{vmatrix}\end{vmatrix} = -\frac{1}{48}\begin{vmatrix}160 & 112\\47 & 38\end{vmatrix} = -\frac{1}{48}\left|816\right| = -17.$$

Take the problem of computing

$$\begin{vmatrix}1 & 2 & 3 & 4 & 5\\3 & 7 & 2 & 1 & 1\\9 & 4 & 1 & 2 & 3\\8 & 1 & 3 & 7 & 2\\4 & 2 & 0 & 3 & 1\end{vmatrix}.$$

We work as follows.

$$\begin{vmatrix}1 & 2 & 3 & 4 & 5\\3 & 7 & 2 & 1 & 1\\9 & 4 & 1 & 2 & 3\\8 & 1 & 3 & 7 & 2\\4 & 2 & 0 & 3 & 1\end{vmatrix} = \frac{1}{3\times 9\times 8}\begin{vmatrix}1 & -7 & -11 & -14\\-51 & -15 & -3 & 0\\-23 & 19 & 47 & -6\\12 & -12 & -4 & 0\end{vmatrix} = \frac{1}{216}\cdot\frac{1}{(-51)\times(-23)}\begin{vmatrix}-372 & -564 & -714\\-1314 & -2466 & 306\\48 & -472 & 72\end{vmatrix}.$$

Further computation gives

$$\begin{vmatrix}1 & 2 & 3 & 4 & 5\\3 & 7 & 2 & 1 & 1\\9 & 4 & 1 & 2 & 3\\8 & 1 & 3 & 7 & 2\\4 & 2 & 0 & 3 & 1\end{vmatrix} = \frac{1}{253368}\cdot\frac{1}{(-1314)}\begin{vmatrix}176256 & -1052028\\738576 & -109296\end{vmatrix} = -\frac{1}{332925552}\left|757738556352\right| = -2276.$$

It is worthy of note that the problem of division by zero which sometimes appear in the condensation technique discussed in this paper can be resolved by applying **Bhaskara's Law of Impending Operation Involving Zero**. See [8], [9], [10], [11], [12] for details.

We take the last instance. Let it be proposed to find the value of the 6th order determinant

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix}.$$

This is performed as follows:

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix} = \frac{1}{1 \times 3 \times (-1) \times 4} \begin{vmatrix} 3 & 3 & 2 & 5 & 8 \\ 0 & -5 & -6 & -3 & -5 \\ 15 & 10 & 0 & 9 & 10 \\ 0 & -13 & -9 & -3 & -2 \\ 36 & 5 & 13 & 11 & 10 \end{vmatrix}.$$

Because of the zeros in the first column of the derived determinant, we interchange columns 1 and 5. Thus, we write

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix} = -\frac{1}{1 \times 3 \times (-1) \times 4} \begin{vmatrix} 8 & 3 & 2 & 5 & 3 \\ -5 & -5 & -6 & -3 & 0 \\ 10 & 10 & 0 & 9 & 15 \\ -2 & -13 & -9 & -3 & 0 \\ 10 & 5 & 13 & 11 & 36 \end{vmatrix}$$

which becomes

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix} = \frac{1}{12} \cdot \frac{1}{(-5) \times 10 \times (-2)} \begin{vmatrix} -25 & -38 & 1 & 15 \\ 0 & 60 & -15 & -75 \\ -110 & -90 & -12 & 30 \\ 120 & 64 & 8 & -72 \end{vmatrix}.$$

Again, to avoid division by zero we interchange columns 1 and 3. So, we write

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix} = \frac{1}{1200} \begin{vmatrix} 1 & -38 & -25 & 15 \\ -15 & 60 & 0 & -75 \\ -12 & -90 & -110 & 30 \\ 8 & 64 & 120 & -72 \end{vmatrix}$$

which becomes

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix} = \frac{1}{1200} \cdot \frac{1}{(-15) \times (-12)} \begin{vmatrix} -510 & -375 & 150 \\ 2070 & 1650 & -1350 \\ -48 & -560 & 624 \end{vmatrix}$$

which in its turn becomes

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix} = \frac{1}{216,000} \cdot \frac{1}{2070} \begin{vmatrix} -65250 & 378000 \\ -1080000 & 1226880 \end{vmatrix}.$$

Finally, we write

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 3 & 6 & 1 & -3 & 6 & 10 \\ -1 & 3 & 3 & 1 & 1 & 0 \\ 4 & -12 & 1 & 5 & -1 & 2 \\ 3 & 0 & 2 & 7 & 2 & 4 \end{vmatrix} = \frac{1}{447,120,000} |328,186,080,000| = 734.$$

3 Application of the New Condensation Method in Solving Linear Systems

In this section, we apply the new condensation method for determining the exact solution set of simultaneous linear equations. We begin by considering the n simultaneous linear equations [6], [13], [14]:

$$\begin{aligned} a_{11,1}x_1 + a_{12,1}x_2 + a_{13,1}x_3 + \cdots + a_{1n,1}x_n &= b_{1,1} \\ a_{21,1}x_1 + a_{22,1}x_2 + a_{23,1}x_3 + \cdots + a_{2n,1}x_n &= b_{2,1} \\ a_{31,1}x_1 + a_{32,1}x_2 + a_{33,1}x_3 + \cdots + a_{3n,1}x_n &= b_{3,1} \\ &\vdots \\ a_{n1,1}x_1 + a_{n2,1}x_2 + a_{n3,1}x_3 + \cdots + a_{nn,1}x_n &= b_{n,1} \end{aligned} \tag{3.1}$$

where x_j are the unknowns, $a_{ij,1}$ are the coefficients of the system (3.1), and $b_{j,1}$ are the constant terms. The application of the knowledge of matrix gives us the matrix form of (3.1):

$$\begin{bmatrix} a_{11,1} & a_{12,1} & a_{13,1} & \cdots & a_{1n,1} \\ a_{21,1} & a_{22,1} & a_{23,1} & \cdots & a_{2n,1} \\ a_{31,1} & a_{32,1} & a_{33,1} & \cdots & a_{3n,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1,1} & a_{n2,1} & a_{n3,1} & \cdots & a_{nn,1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ \vdots \\ b_{n,1} \end{bmatrix}. \tag{3.2}$$

For quick and easy computation , we write the matrix equation (3.2) in augmented form. The 1st augmented matrix equation of (3.1) is,thus:

$$\begin{bmatrix} a_{11,1} & a_{12,1} & a_{13,1} & \cdots & a_{1n,1} & b_{1,1} \\ a_{21,1} & a_{22,1} & a_{23,1} & \cdots & a_{2n,1} & b_{2,1} \\ a_{31,1} & a_{32,1} & a_{33,1} & \cdots & a_{3n,1} & b_{3,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1,1} & a_{n2,1} & a_{n3,1} & \cdots & a_{nn,1} & b_{n,1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}. \tag{3.3}$$

To condense the n simultaneous linear equations to $(n-1)$ simultaneous linear equations, we make $a_{11,1}x_1, a_{21,1}x_1, \ldots, a_{n1,1}x_1$ the subjects of the linear equations (3.1) and obtain the following results:

$$\begin{aligned} a_{11,1}x_1 &= b_{1,1} - a_{12,1}x_2 - a_{13,1}x_3 - \cdots - a_{1n,1}x_n \\ a_{21,1}x_1 &= b_{2,1} - a_{22,1}x_2 - a_{23,1}x_3 - \cdots - a_{2n,1}x_n \\ a_{31,1}x_1 &= b_{3,1} - a_{32,1}x_2 - a_{33,1}x_3 - \cdots - a_{3n,1}x_n \\ &\vdots \\ a_{n1,1}x_1 &= b_{n,1} - a_{n2,1}x_2 - a_{n3,1}x_3 - \cdots - a_{nn,1}x_n. \end{aligned} \tag{3.4}$$

We then divide each equation of (3.4) by the equation directly under it and have the following $(n-1)$ equations:

$$\begin{aligned} \frac{a_{11,1}}{a_{21,1}} &= \frac{b_{1,1} - a_{12,1}x_2 - a_{13,1}x_3 - \cdots - a_{1n,1}x_n}{b_{2,1} - a_{22,1}x_2 - a_{23,1}x_3 - \cdots - a_{2n,1}x_n} \\ \frac{a_{21,1}}{a_{31,1}} &= \frac{b_{2,1} - a_{22,1}x_2 - a_{23,1}x_3 - \cdots - a_{2n,1}x_n}{b_{3,1} - a_{32,1}x_2 - a_{33,1}x_3 - \cdots - a_{3n,1}x_n} \\ &\vdots \\ \frac{a_{(n-1)1,1}}{a_{n1,1}} &= \frac{b_{(n-1),1} - a_{(n-1)2,1}x_2 - a_{(n-1)3,1}x_3 - \cdots - a_{(n-1)n,1}x_n}{b_{n,1} - a_{n2,1}x_2 - a_{n3,1}x_3 - \cdots - a_{nn,1}x_n}. \end{aligned} \tag{3.5}$$

Simplifying each of the equations in (3.5), we get

$$\begin{aligned} &(a_{11,1}a_{22,1} - a_{12,1}a_{21,1})x_2 + (a_{11,1}a_{23,1} - a_{13,1}a_{21,1})x_3 + \ldots \\ &\quad +(a_{11,1}a_{2n,1} - a_{1n,1}a_{21,1})x_n = a_{11,1}b_{2,1} - a_{21,1}b_{1,1} \\ \\ &(a_{21,1}a_{32,1} - a_{22,1}a_{31,1})x_2 + (a_{21,1}a_{33,1} - a_{23,1}a_{31,1})x_3 + \ldots \\ &\quad +(a_{21,1}a_{3n,1} - a_{2n,1}a_{31,1})x_n = a_{21,1}b_{3,1} - a_{31,1}b_{2,1} \\ &\vdots \\ &(a_{(n-1),1}a_{n2,1} - a_{(n-1)2,1}a_{n1,1})x_2 + (a_{(n-1)1,1}a_{n3,1} - a_{(n-1)3,1}a_{n1,1})x_3 + \ldots \\ &\quad +(a_{(n-1)1,1}a_{nn,1} - a_{(n-1)n,1}a_{n1,1})x_n = a_{(n-1)1,1}b_{n,1} - a_{n1,1}b_{(n-1),1} \end{aligned} \tag{3.6}$$

The coefficients of $x_2, x_3, \ldots, x_n$ in (3.6) are put in determinant form as shown below:

$$\begin{aligned} &\begin{vmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \end{vmatrix} x_2 + \begin{vmatrix} a_{11,1} & a_{13,1} \\ a_{21,1} & a_{23,1} \end{vmatrix} x_3 + \cdots + \begin{vmatrix} a_{11,1} & a_{1n,1} \\ a_{21,1} & a_{2n,1} \end{vmatrix} x_n = \begin{vmatrix} a_{11,1} & b_{1,1} \\ a_{21,1} & b_{2,1} \end{vmatrix} \\ &\begin{vmatrix} a_{21,1} & a_{22,1} \\ a_{31,1} & a_{32,1} \end{vmatrix} x_2 + \begin{vmatrix} a_{21,1} & a_{23,1} \\ a_{31,1} & a_{33,1} \end{vmatrix} x_3 + \cdots + \begin{vmatrix} a_{21,1} & a_{2n,1} \\ a_{31,1} & a_{3n,1} \end{vmatrix} x_n = \begin{vmatrix} a_{21,1} & b_{2,1} \\ a_{31,1} & b_{3,1} \end{vmatrix} \\ &\vdots \\ &\begin{vmatrix} a_{(n-1)1,1} & a_{(n-1)2,1} \\ a_{n1,1} & a_{n2,1} \end{vmatrix} x_2 + \begin{vmatrix} a_{(n-1)1,1} & a_{(n-1)3,1} \\ a_{n1,1} & a_{n3,1} \end{vmatrix} x_3 + \ldots + \begin{vmatrix} a_{(n-1)1,1} & a_{(n-1)n,1} \\ a_{n1,1} & a_{nn,1} \end{vmatrix} x_n \\ &\quad = \begin{vmatrix} a_{(n-1)1,1} & b_{(n-1),1} \\ a_{n1,1} & b_{n,1} \end{vmatrix}. \end{aligned} \tag{3.7}$$

The equations (3.7) are $(n-1)$ simultaneous linear equations with $(n-1)$ unknowns: $x_2, x_3, \ldots, x_n$. The n simultaneous linear equations (3.1) is said to have been condensed to the $(n-1)$ simultaneous linear equations (3.7) . Expressing (3.7) in augmented matrix form, we get the 2nd augmented matrix equation with $(n-1)$ unknowns:

$$\begin{bmatrix} \begin{vmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \end{vmatrix} & \begin{vmatrix} a_{11,1} & a_{13,1} \\ a_{21,1} & a_{23,1} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11,1} & b_{1,1} \\ a_{21,1} & b_{2,1} \end{vmatrix} \\ \begin{vmatrix} a_{21,1} & a_{22,1} \\ a_{31,1} & a_{32,1} \end{vmatrix} & \begin{vmatrix} a_{21,1} & a_{23,1} \\ a_{31,1} & a_{33,1} \end{vmatrix} & \cdots & \begin{vmatrix} a_{21,1} & b_{2,1} \\ a_{31,1} & b_{3,1} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{(n-1)1,1} & a_{(n-1)2,1} \\ a_{n1,1} & a_{n2,1} \end{vmatrix} & \begin{vmatrix} a_{(n-1)1,1} & a_{(n-1)3,1} \\ a_{n1,1} & a_{n3,1} \end{vmatrix} & \cdots & \begin{vmatrix} a_{(n-1)1,1} & b_{(n-1),1} \\ a_{n1,1} & b_{n,1} \end{vmatrix} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ \cdots \\ x_n \end{bmatrix}.$$

If the 2nd augmented matrix equation is written as

$$\begin{bmatrix} a_{11,2} & a_{12,2} & a_{13,2} & \cdots & a_{1(n-1),2} & b_{1,2} \\ a_{21,2} & a_{22,2} & a_{23,2} & \cdots & a_{2(n-1),2} & b_{2,2} \\ a_{31,2} & a_{32,2} & a_{33,2} & \cdots & a_{3(n-1),2} & b_{3,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)1,2} & a_{(n-1)2,2} & a_{(n-1)3,2} & \cdots & a_{(n-1)(n-1),2} & b_{(n-1),2} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \cdots \\ x_n \end{bmatrix}$$

then

$$a_{ij,2} = \begin{vmatrix} a_{i1,1} & a_{i(j+1),1} \\ a_{(i+1)1,1} & a_{(i+1)(j+1),1} \end{vmatrix}$$

and

$$b_{i,2} = \begin{vmatrix} a_{i1,1} & b_{i,1} \\ a_{(i+1)1,1} & b_{(i+1),1} \end{vmatrix}$$

where $a_{ij,2}$ is the element in row i and column j before the last column of the 2nd augmented matrix and $b_{i,2}$ is the constant term in row i and the last column of the 2nd augmented matrix.

The method is repeated until the nth augmented matrix equation of the form

$$\begin{bmatrix} a_{11,n} & b_{1,n} \end{bmatrix} \begin{bmatrix} x_n \end{bmatrix}$$

is obtained. This is a linear equation whose solution is

$$x_n = \frac{b_{1,n}}{a_{11,n}}.$$

The value of x_n is substituted back into the $(n-1)$ th augmented matrix equation to obtain x_{n-1}. The back-substitution is repeated to obtain the other unknowns: $x_{n-2}, x_{n-3}, x_{n-4}, \ldots, x_1$.

To make this method clear, let's consider some examples. Let us solve the linear system

$$\begin{aligned} 2x_1 + 3x_2 &= 8 \\ 3x_1 - x_2 &= 1 \end{aligned}.$$

The augmented matrix form is

$$\begin{bmatrix} 2 & 3 & 8 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This is condensed into

$$\begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 8 \\ 3 & 1 \end{vmatrix} \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix}$$

which gives

$$\begin{bmatrix} -11 & -22 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix}.$$

Thus,

$$x_2 = \frac{-22}{-11} = 2.$$

From the 1st row of the augmented matrix equation, we have

$$2x_1 + 3x_2 = 8.$$

Substituting the value of x_2 into the equation, we get

$$\begin{aligned} 2x_1 + 3(2) &= 8 \\ 2x_1 + 6 &= 8 \\ 2x_1 &= 2 \\ x_1 &= 1. \end{aligned}$$

Therefore, the solution set of the simultaneous linear equations is (1,2).

We now take the linear system

$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 2 \\ 2x_1 - 3x_2 - 4x_3 &= 5. \\ 3x_1 + x_2 - x_3 &= 8 \end{aligned}$$

To solve this, we state with the 1st augmented matrix equation

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & -3 & -4 & 5 \\ 3 & 1 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Condensing the equation to the 2nd augmented matrix equation, we get

$$\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}.$$

Evaluating the determinants, we have the 2nd augmented matrix equation as

$$\begin{bmatrix} -9 & -10 & 1 \\ 11 & 10 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}.$$

Condensing the 2nd augmented matrix equation to the 3rd augmented matrix equation, we get

$$\begin{bmatrix} \begin{vmatrix} -9 & -10 \\ 11 & 10 \end{vmatrix} & \begin{vmatrix} -9 & 1 \\ 11 & 1 \end{vmatrix} \end{bmatrix} \begin{bmatrix} x_3 \end{bmatrix}.$$

We evaluate the determinants and have

$$\begin{bmatrix} 20 & -20 \end{bmatrix} \begin{bmatrix} x_3 \end{bmatrix}.$$

Thus,

$$x_3 = -\frac{20}{20} = -1.$$

From the 1st row of the 2nd augmented matrix equation, we obtain the equation

$$-9x_2 - 10x_3 = 1.$$

Substituting the value of x_3 into the equation above we get

$$-9x_2 - 10(-1) = 1.$$

Therefore,

$$x_2 = 1.$$

From the 1st row of the 1st augmented matrix equation, we obtain the equation

$$x_1 + 3x_2 + 3x_3 = 2.$$

Substituting the values of x_2 and x_3 into the equation above, we have

$$x_1 + 3(1) + 3(-1) = 2.$$

Therefore,

$$x_1 = 2.$$

Thus, the solution set of the simultaneous linear equations is $(2, 1, -1)$.

We wish to solve the linear system

$$\begin{aligned} x_1 + 3x_2 + 5x_3 - x_4 &= 1 \\ 4x_2 - x_3 + 2x_4 &= 11 \\ x_1 - x_2 + 5x_3 - x_4 &= -7 \\ 3x_1 + x_2 - 3x_3 + x_4 &= 9. \end{aligned}$$

The 1st augmented matrix equation is

$$\begin{bmatrix} 1 & 3 & 5 & -1 & 1 \\ 0 & 4 & -1 & 2 & 11 \\ 1 & -1 & 5 & -1 & -7 \\ 3 & 1 & -3 & 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

If we work with the zero in the 1st column, we will obtain a wrong solution set. This is so because if we are to find the determinant of the matrix using the condensation method, we will get division by zero. In the previous section we gave ways of overcoming it, namely elementary row operations. Here, however, we apply only the following elementary row operation: add a multiple of one row to another. Therefore, if we add row 1 to row 2, we get

$$\begin{bmatrix} 1 & 3 & 5 & -1 & 1 \\ 1 & 7 & 4 & 1 & 12 \\ 1 & -1 & 5 & -1 & -7 \\ 3 & 1 & -3 & 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Condensing the equation to the 2nd augmented matrix equation, we get

$$\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 12 \end{vmatrix} \\ \begin{vmatrix} 1 & 7 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 12 \\ 1 & -7 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 3 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -7 \\ 3 & 9 \end{vmatrix} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} 4 & -1 & 2 & 11 \\ -8 & 1 & -2 & -19 \\ 4 & -18 & 4 & 30 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

We condense this 2nd augmented matrix equation to the 3rd augmented matrix equation,

$$\begin{bmatrix} \begin{vmatrix} 4 & -1 \\ -8 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 2 \\ -8 & -2 \end{vmatrix} & \begin{vmatrix} 4 & 11 \\ -8 & -19 \end{vmatrix} \\ \begin{vmatrix} -8 & 1 \\ 4 & -18 \end{vmatrix} & \begin{vmatrix} -8 & -2 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} -8 & -19 \\ 4 & 30 \end{vmatrix} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

This becomes

$$\begin{bmatrix} -4 & 8 & 12 \\ 140 & -24 & -164 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

Finally, we condense the 3rd augmented matrix equation to the 4th augmented matrix equation. We get

$$\begin{bmatrix} \begin{vmatrix} -4 & 8 \\ 140 & -24 \end{vmatrix} & \begin{vmatrix} -4 & 12 \\ 140 & -164 \end{vmatrix} \end{bmatrix} \begin{bmatrix} x_4 \end{bmatrix}.$$

Evaluating the determinants, we get

$$\begin{bmatrix} -1024 & -1024 \end{bmatrix} \begin{bmatrix} x_4 \end{bmatrix}.$$

Therefore, the value of x_4 is

$$x_4 = \frac{-1024}{-1024} = 1.$$

From the 1st row of the 3rd augmented matrix equation, we obtain the equation

$$-4x_3 + 8x_4 = 12.$$

Substituting the value of x_4 into the equation above, we have

$$\begin{aligned} -4x_3 + 8(1) &= 12 \\ x_3 &= -1. \end{aligned}$$

From the the 1st row of the 2nd augmented matrix equation, we obtain the equation

$$4x_2 - x_3 + 2x_4 = 11.$$

Substituting the values of x_3 and x_4 into the equation above, we have

$$\begin{aligned} 4x_2 - (-1) + 2(1) &= 11 \\ x_2 &= 2. \end{aligned}$$

From the the 1st row of the 1st augmented matrix equation, we obtain the equation

$$x_1 + 3x_2 + 5x_3 - x_4 = 1.$$

Substituting the values of x_2, x_3 and x_4 into the equation above, we have

$$\begin{aligned} x_1 + 3(2) + 5(-1) - 1 &= 1 \\ x_1 &= 1. \end{aligned}$$

Therefore, the solution set of the simultaneous equations is (1,2,-1,1).

We take the last instance. Consider the linear system

$$\begin{aligned} 2x_1 + x_2 - 4x_3 - 7x_4 + x_5 &= -18 \\ x_1 - x_2 + 4x_3 + x_4 - x_5 &= 18 \\ 3x_1 + 2x_2 - x_3 - x_4 + x_5 &= -2 \\ x_1 - x_2 + 5x_3 + 6x_4 - 5x_5 &= 28 \\ -x_1 + 3x_2 - x_3 - x_4 + x_5 &= -11 \end{aligned}$$

The method of solving the equations is the same as in the previous examples. So we will omit the descriptive statement for every step. We begin with

$$\begin{bmatrix} 2 & 1 & -4 & -7 & 1 & -18 \\ 1 & -1 & 4 & 1 & -1 & 18 \\ 3 & 2 & -1 & -1 & 1 & -2 \\ 1 & -1 & 5 & 6 & -2 & 28 \\ -1 & 3 & -1 & -1 & 1 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

So, we have

$$\begin{bmatrix} -3 & 12 & 9 & -3 & 54 \\ 5 & -13 & -4 & 4 & -56 \\ -5 & 16 & 19 & -7 & 86 \\ 2 & 4 & 5 & -1 & 17 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} -21 & -33 & 3 & -102 \\ 15 & 75 & -15 & 150 \\ -52 & -63 & 19 & -257 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} -1080 & 270 & -1620 \\ 2955 & -495 & 3945 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} -263250 & 526500 \end{bmatrix} \begin{bmatrix} x_5 \end{bmatrix}$$

Therefore, we obtain

$$\begin{aligned} x_5 &= \frac{526500}{-263250} = -2 \\ x_4 &= \frac{-1620 - 270x_5}{-1080} = \frac{-1620 - 270(-2)}{-1080} = 1 \\ x_3 &= \frac{-102 - 3x_5 + 33x_4}{-21} = \frac{-102 - 3(-2) + 33(1)}{-21} = 3 \\ x_2 &= \frac{54 + 3x_5 - 9x_4 - 12x_3}{-3} = \frac{54 + 3(-2) - 9(1) - 12(3)}{-3} = -1 \\ x_1 &= \frac{-18 - x_5 + 7x_4 + 4x_3 - x_2}{2} = \frac{-18 - (-2) + 7(1) + 4(3) - (-1)}{2} = 2. \end{aligned}$$

The solution set is (2,-1,3,1,-2).

Conclusion

In this paper we discussed a novel condensation method of determinant evaluation, illustrating it with sufficient instances. We closed our discussion with the application of the method in determining the solution sets of systems of linear equations.

Acknowledgment

My completion of this work could not have been accomplished without the continued support and encouragement of Dr. Agun Ikhile.

Competing Interests

Author has declared that no competing interests exist.

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