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Asymptotics of Solution of a Boundary Value Problem for Quasilinear Non-Classical Type Differential Equation of Arbitrary Odd Order

Mahir M. Sabzaliev[1*] and Ilhama M. Sabzalieva[1]

[1]*Azerbaijan State Oil and Industrial University, Az 1026, 20, Azadliq Ave, Baku, Azerbaijan.*

Authors' contributions

This work was carried out in collaboration between both authors. Author MMS introduced the original problem. Author IMS carried out the iteration process and estimated the remainder term. The formulation of the main result was conducted by Mahir M. Sabzaliev. Both authors read and approved the final manuscript.

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Abstract

In a rectangle domain, a boundary value problem is considered for a singularly perturbed quasilinear non-classical type equation of arbitrary odd order, degenerating into a hyperbolic equation. Asymptotic expansion of the generalized solution of the problem under consideration is constructed to within any positive degree of a small parameter, and the residual term is estimated.

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Corresponding author: E-mail: sabzalievm@mail.ru;

1 Introduction and Problem Statement

When studying numerical phenomena where there are nonuniform transitions from one physical characteristics to another ones, it is necessary to study singularly perturbed boundary value problems (see e.g. [1]-[4]). Singularly perturbed problems were first studied from different positions by A.N. Tikhonov [5], L.S. Pontryagin [6], I. Vishik and Lusternik [7], [8], Vazov [9], S.A. Lomov [10], A.M. Ilin [11], and other scientists.

Theory of singularly perturbed boundary value problems for linear partial equations was significantly developed in M.I. Vishik's and L.A. Lusternik's papers [7, 8]. After appear ance of these papers, this method was generalized both by the followers of M.I. Vishik and A.A. Lusternik, and by other researchers. However, all of studied boundary value problems were related to one of three classical types.

In [7, §8] M.I. Vishik and L.A. Lusternik introduced the so called one-characteristic linear equations that are not classical equations. They called the $2k+1$ odd order equation of the form $A_1\left(A_{2k}u\right)+B_{2k}u=f$ one-characteristic if A_1 is a first order operator, A_{2k} is an elliptic operator of order $2k$, while B_{2k} is any differential operator of order at most $2k$. In the paper [7], they studied mutual degenerations of one-characteristic and elliptic equations. In this paper, in $D=\{(x,y)\,|0\le x\le 1,\ 0\le y\le 1\}$ they considered the boundary value problem

$$\varepsilon^2\frac{\partial}{\partial x}\left(\Delta u\right)-\varepsilon\Delta u+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+u=f\left(x,y\right), \tag{1.1}$$

$$u|_\Gamma=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=1}=0, \tag{1.2}$$

where $\varepsilon>0$ is a small parameter, $\Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator, $f\left(x,y\right)$ is the given function, Γ is the boundary of the domain D. Assuming that $f\left(x,y\right)$ for $x=y$ together with its derivatives of appropriate order vanishes, they constructed only the first terms of the asymptotic solution of boundary value problem (1.1), (1.2).

In the paper [12], M.G. Javadov and M.M. Sabzaliev rejecting from the condition of vanishing of the function $f\left(x,y\right)$ for $x=y$ constructed first members of the asymptotic solution of boundary value problem (1.1), (1.2) allowing for inner layers arising near $x=y$. In this paper, the complete asymptotics of the solution of a boundary value problem for equation (1.1) in the infinite strip $\Pi=\{\,(x,y)|\,0\le x\le 1,-\infty<y<+\infty\}$ was also constructed. Complete asymptotics of the solution of boundary value problem (1.1), (1.2) was constructed by Sabzaliev M.M. in [13].

Degeneration of an one-characteristic equation into an elliptic one was studied in the paper [7] on the following boundary value problem on a rectangle D:

$$\frac{\partial}{\partial x}\left(\Delta u\right)-\Delta u=f\left(x,y\right),\quad u|_\Gamma=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=1}=0.$$

Complete asymptotics in a small parameter of the solution of a boundary value problem in the infinite strip $\Pi=\{\,(x,y)|\,0\le x\le 1,-\infty<y<+\infty\}$ for the equation

$$\varepsilon\frac{\partial}{\partial x}\left(\Delta u\right)-\Delta u+au=f\left(x,y\right)\ ,\ \ (a=const>0)$$

was constructed by M.M. Sabzaliev in [14]. M.M. Sabzaliev and M.E. Kerimova studied degeneration of one-characteristic equation into parabolic one. In the paper [15], in a rectangle $D=\{\,(t,x)|\,0\le t\le T,0\le x\le 1\}$ they considered the following boundary value problem

$$\varepsilon^2\frac{\partial}{\partial x}\left(\Delta u\right)-\varepsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+au=f\left(t,x\right), \tag{1.3}$$

$$u|_{t=0} = 0, \quad u|_{t=T} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=T} = 0, \ \ (0 \le x \le 1) \tag{1.4}$$

$$u|_{x=0} = u|_{x=1} = 0, \ \ (0 \le t \le T) \tag{1.5}$$

where $\Delta \equiv \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$, $a = const > 0$. Complete asymptotics of the solution of boundary value problem (1.3)-(1.5) was constructed. In the paper [16] a boundary value problem was studied for equation (1.3) in the semi-infinite strip $\Pi_+ = \{ (t,x) |\, 0 \le t \le 1,\, 0 \le x < +\infty\}$. In this case boundary conditions (1.4) remain and instead of conditions in (1.5), the followings are considered: $u|_{x=0} = 0$, $\lim\limits_{x \to +\infty} u = 0$. A boundary value problem in the infinite strip

$$\Pi = \{ (t,x) |\, 0 \le t \le 1, -\infty < x < +\infty\}$$

for equation (1.3) was considered in [17]. Here, instead of condition (1.5) the boundary conditions $\lim\limits_{|x| \to +\infty} u = 0$ are taken.

Some singularly perturbed linear equations of non-classical type were researched by Ya. Sh. Salimov and I.M. Sabzalieva in the papers [18]-[22].

The all above-mentioned studies are related to linear differential equations of non-classical type. The Vishik-Lusternik technique for constructing asymptotics in a small parameter of solutions of boundary value problems for linear and differential equations are taken to some classes of nonlinear differential equations as well. However, study of nonlinear singularly perturbed boundary value problems by this technique is accompanied by bulky calculations. In the paper [23], M.I. Vishik and L.A. Lusternik illustrated the technique for constructing nonlinear differential equations on the following boundary value problem:

$$\varepsilon y'' + \varphi(x,y)\, y' - \psi(x,y)\,, \ \ y(0) = A, \ \ y(1) = B.$$

Asymptotic of the solution of this problem in powers of parameters A was studied by V. Vazov.

In [24] Su Yui Chen constructed asymptotics in small parameter of the solution of a mixed problem for the quasilinear hyperbolic equation

$$\varepsilon \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right) - \varphi(t,x,u) \frac{\partial u}{\partial t} + \psi(t,x,u) = 0.$$

In [25], Trenogin set up asymptotics in a small parameter of the solution of the boundary value problem

$$\frac{\partial u}{\partial t} - \varepsilon b(x,t) \frac{\partial^2 u}{\partial x^2} + c(x,t,u) = 0, \ \ u(x,0) = \varphi(x)\,, \ \ u(0,t) = u(l,t) = 0.$$

In [26], V.Yu. Lunin constructed complete asymptotics of the solution of the Dirichlet problem for the nonlinear elliptic equation

$$-\varepsilon^4 \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right)^3 - \varepsilon^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + F(x,u) = 0.$$

We also note the papers [27]-[32] that were devoted to construction of asymptotics of solutions of different boundary value problems for singularly perturbed nonlinear differential equations.

The carried out analysis represents the following characteristics of the studied singularly perturbed nonlinear boundary value problems. In the first place, a great majority of singularly perturbed nonlinear equations degenerate for $\varepsilon = 0$ either into functional or ordinary differential equations. In the studied equations the derivatives of the sought-for function enter into the equation linearly, and only the sought-for function itself enters non-linearly into the equation. The domains under consideration are bounded and have no viscous adjoint boundaries (under viscous boundary we understand a boundary in the vicinity of which there arises a boundary layer). Finally, all studied singularly perturbed nonlinear partial differential equations relate to one three classical types.

Note some papers [33]-[38] of the first author of this paper, where singularly perturbed quasilinear elliptic and hyperbolic equations are studied in bounded and in unbounded domains, and under degeneration we get partial equations and the derivatives of the sought-for function enter into the equation non-linearly.

In the present paper we consider a boundary value problem for a non-classical type singularly perturbed quasilinear equation of arbitrary odd order degenerating into a hyperbolic equation. The domains where this boundary value problem is studied, has three adjoining viscous boundaries.

In $D = \{ (t,x) | \, 0 \le t \le 1, \;\; 0 \le x \le 1\}$ we consider the following boundary value problem

$$L_\varepsilon u \equiv (-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} u}{\partial t^{2m+1}} - \varepsilon^p \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p - \varepsilon \frac{\partial^2 u}{\partial x^2} +$$

$$+ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + a(t,x)\, u - f(t,x) = 0, \tag{1.6}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = ... = \left.\frac{\partial^m u}{\partial t^m}\right|_{t=0} = 0, \;\; (0 \le x \le 1) \tag{1.7}$$

$$\left.\frac{\partial^{m+1} u}{\partial t^{m+1}}\right|_{t=1} = \left.\frac{\partial^{m+2} u}{\partial t^{m+2}}\right|_{t=1} = ... = \left.\frac{\partial^{2m} u}{\partial t^{2m}}\right|_{t=1} = 0, \;\; (0 \le x \le 1) \tag{1.8}$$

$$u|_{x=0} = 0, \quad u|_{x=1} = 0, \quad (0 \le t \le 1) \tag{1.9}$$

where $p = 2k + 1,\ k$ and m are arbitrary natural numbers, $a(t,x) \ge \gamma^2 > 0$ and $f(t,x)$ are the smooth functions given in D.

In this paper, our goal is to construct asymptotic expansion of the solution of boundary value problem (1.6)-(1.9). When constructing asymptotics we are guided by the Vishik-Lusternik technique. It should be noted that it is not succeeded to construct asymptotics of the solution of the problem under consideration by traditional way. In this connection, here the first iterative process and the iterative process that helps to construct boundary layer type functions, are embedded one to another. Furthermore, the notion "satisfaction of boundary condition approximately to within any positive degree of a small parameter" was introduced. In what follows, this notion is used when estimating the residual term. Note that earlier such an approach was used by the first author of this paper in the papers [35], [36].

2 Carrying Out Iterative Processes

In the first iterative process, the approximate solution of equation (1.6) is sought in the form

$$W = W_0(t,x) + \varepsilon W_1(t,x) + ... + \varepsilon^n W_n(t,x), \tag{2.1}$$

and the functions $W_i(t,x);\ i = 0, 1, ..., n$ will be chosen so that

$$L_\varepsilon W = 0(\varepsilon^{n+1}). \tag{2.2}$$

Substituting (2.1) in (2.2), expanding the nonlinear term in powers of ε and grouping the terms with identical powers of ε, for determining $W_i; i = 0, 1, ..., n$ we get the following recurrently connected equations

$$\frac{\partial W_0}{\partial t} + \frac{\partial W_0}{\partial x} + aW_0 = f(t, x), \tag{2.3}$$

$$\frac{\partial W_i}{\partial t} + \frac{\partial W_i}{\partial x} + aW_i = f_i\left(t, x\right), \tag{2.4}$$

where $f_i\left(t, x\right)$ are the known functions dependent on $W_0, W_1, ..., W_{i-1};\ i = 1, 2, ..., n$ and their derivatives. Here we give formulas only for $f_1\left(t, x\right)$: $f_1\left(t, x\right) = \frac{\partial^2 W_0}{\partial x^2}$.

For equations (2.3), (2.4) with respect to x we should use the first condition from (1.9), i.e.

$$W\ _i|_{x=0} = 0,\ i = 0, 1, ..., n. \tag{2.5}$$

Boundary conditions with respect to t for equations (2.3), (2.4) is given below. Now we note that with respect to t we will use the first condition from (1.8) for $t = 0$. Under such a choice of boundary conditions for equations (2.3), (2.4) on the boundary $S_1 = \{\,(t, x)|\, t = 0,\ \ 0 \le x \le 1\}$, m conditions from $m + 1$ boundary conditions of (1.7), on the boundary $S_2 = \{\,(t, x)|\, t = 1,\ \ 0 \le x \le 1\}$ all m conditions of (1.8), and on the boundary $S_3 = \{\,(t, x)|\ 0 \le t \le 1,\ \ x = 1\}$ the second condition of (1.9) will be lost. For compensating the lost boundary conditions we should construct boundary layer functions near the boundaries S_1, S_2, S_3.

If as usually for all functions W_i entering into expansion $W = \sum\limits_{i=0}^{n} \varepsilon^i W_i$ we use the condition $W_i|_{t=0} = 0$ and carry out first iterative process and then construct the boundary layer type function $V = \sum\limits_{j=0}^{n_1} \varepsilon^{1+j} V_j$ near the boundary S_1, for compensating the lost conditions $\frac{\partial^k}{\partial t^k}\left(W + V\right)\Big|_{t=0} = 0$; $k = 1, 2, ..., m$, then the sum $W + V$ does not satisfy the boundary condition $\left(W + V\right)|_{t=0} = 0$ which the function W satisfied. To overcome these difficulties, the first iterative process by means of which the functions W_i are constructed, and the iterative process that helps to construct the boundary layer type function V_j near the boundary S_1, are embedded one to another. Therefore before finding boundary conditions with respect to t, for equations (2.3), (2.4) at first it is necessary to obtain the equations whose solutions will be boundary layer functions near S_1.

We should look for the boundary layer function V near the boundary S_1, in the form

$$V = \varepsilon\left(V_0 + \varepsilon V_1 + ... + \varepsilon^{n_1} V_{n_1}\right), \tag{2.6}$$

as the solution of the equation

$$L_{\varepsilon,1}\left(W + V\right) - L_{\varepsilon,1} W = 0\left(\varepsilon^{n_1+1}\right), \tag{2.7}$$

where $L_{\varepsilon,1}$ is a new decomposition of the operator L_ε near the boundary S_1 in the coordinates (ξ, x), where $\xi = \frac{t}{\varepsilon}$. The exact value of n_1 will be determined later. Now we only note that $n_1 \ge n + 1$.

From (2.6) and (2.7) we get the following recurrently connected equations for determining the functions $V_j; j = 0, 1, ..., n_1$:

$$(-1)^m \frac{\partial^{2m+1} V_0}{\partial \xi^{2m+1}} + \frac{\partial V_0}{\partial \xi} = 0, \tag{2.8}$$

$$(-1)^m \frac{\partial^{2m+1} V_j}{\partial \xi^{2m+1}} + \frac{\partial V_j}{\partial \xi} = h_j;\ j = 1, 2, ..., n_1, \tag{2.9}$$

where h_j are the known functions dependent on $V_0, V_1, ..., V_{j-1}$ and their derivatives. We give a formula for $h_1\left(\xi, x\right) = -\frac{\partial V_0}{\partial x} - a\left(0, x\right) V_0\left(\xi, x\right)$.

The boundary conditions for equations (2.8), (2.9) whose solutions will be the functions $V_0, V_1, ..., V_{n_1}$, are obtained from the requirement that the sum $W + V$ for $t = 0$ should satisfy conditions of (1.7) except the first condition, i.e.

$$\frac{\partial}{\partial t}(W+V)|_{t=0} = \frac{\partial^2}{\partial t^2}(W+V)|_{t=0} = ... = \frac{\partial^m}{\partial t^m}(W+V)|_{t=0} = 0. \qquad (2.10)$$

Note that (2.10) yields that the number n_1, in (2.6) should be taken equal to $n + m - 1$. In other words, it is necessary to look for the function V in the form

$$V = \varepsilon\left(V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + ... + \varepsilon^{n+m-1} V_{n+m-1}\right). \qquad (2.11)$$

Now we find boundary conditions with respect to t for equations (2.3), (2.4). For that we substitute the expressions (2.1), (2.11) for W, V in the equality

$$(W+V)|_{t=0} = 0 \qquad (2.12)$$

and equate to zero the coefficients for ε whose powers are less than $n + 1$. Then we have

$$W_0|_{t=0} = 0, \quad W_i|_{t=0} = -V_{i-1}|_{\xi=0}; \ \ i = 1, 2, ..., n. \qquad (2.13)$$

It should be noted that if the functions W_i; $i = 0, 1, ..., n$ satisfy condition (2.13), then for the sum $W + V$ boundary condition (2.12) will be satisfied not exactly but approximately, to within ε^{n+1}, i.e.

$$(W+V)|_{t=0} = \varepsilon^{n+1}\varphi_\varepsilon(x). \qquad (2.14)$$

Here the function $\varphi_\varepsilon(x)$ is determined by the formula

$$\varphi_\varepsilon(x) = \left(V_n + \varepsilon V_{n+1} + ... + \varepsilon^{m-1} V_{n+m-1}\right)|_{\xi=0}. \qquad (2.15)$$

Having substituted the expressions for W and V, respectively from (2.1), (2.11) in (2.10), and comparing the terms at identical powers of ε, we get the following boundary conditions for equations (2.8), (2.9)

$$\left.\frac{\partial V_0}{\partial \xi}\right|_{\xi=0} = -\left.\frac{\partial W_0}{\partial t}\right|_{t=0}, \ \left.\frac{\partial^2 V_0}{\partial \xi^2}\right|_{\xi=0} = 0, \ \left.\frac{\partial^3 V_0}{\partial \xi^3}\right|_{\xi=0} = 0, ..., \ \left.\frac{\partial^m V_0}{\partial \xi^m}\right|_{\xi=0} = 0, \qquad (2.16_0)$$

$$\left.\frac{\partial V_1}{\partial \xi}\right|_{\xi=0} = -\left.\frac{\partial W_1}{\partial t}\right|_{t=0}, \ \left.\frac{\partial^2 V_1}{\partial \xi^2}\right|_{\xi=0} = \left.\frac{\partial^2 W_0}{\partial t^2}\right|_{t=0},$$

$$\left.\frac{\partial^3 V_1}{\partial \xi^3}\right|_{\xi=0} = 0, ..., \ \left.\frac{\partial^m V_1}{\partial \xi^m}\right|_{\xi=0} = 0, \qquad (2.16_1)$$

$$. \ ,$$

$$\left.\frac{\partial V_{n+m-1}}{\partial \xi}\right|_{\xi=0} = 0, \ \left.\frac{\partial^2 V_{n+m-1}}{\partial \xi^2}\right|_{\xi=0} = 0, ..., \ \left.\frac{\partial^{m-1} V_{n+m-1}}{\partial \xi^{m-1}}\right|_{\xi=0} = 0,$$

$$\left.\frac{\partial^m V_{n+m-1}}{\partial \xi^m}\right|_{\xi=0} = -\left.\frac{\partial^m W_n}{\partial t^m}\right|_{t=0}. \qquad (2.16_{n+m-1})$$

From (2.13) and (2.16_0)-(2.16_{n+m-1}) it is seen that the functions W_i and V_j will be constructed in turn in the sequence $W_0, V_0, W_1, V_1, ..., W_n, V_n, V_{n+1}, ..., V_{n+m-1}$. Now construct the functions $W_0, W_1, W_n, ..., W_n$ and $V_0, V_1, ..., V_{n+m-1}$. From (2.3), (2.5) for $i = 0$ and (2.13) we have that W_0 is the solution of the following boundary value problem

$$\frac{\partial W_0}{\partial t} + \frac{\partial W_0}{\partial x} + a\left(t,x\right)W_0 = f\left(t,x\right); \;\; W\Big|_{t=0} = 0, \quad W_0|_{x=0} = 0. \tag{2.17}$$

Boundary value problem (2.17) is called a degenerated problem corresponding to problem (1.6)-(1.9). It should be noted that here the solution of the degenerated problem has breaks on the bisectrix $t = x$ of the first quadrant.

The following statement is valid.

Lemma 2.1. *Let $f\left(t,x\right) \in C^{2n+2m+2}\left(D\right)$, $a\left(t,x\right) \in C^{2n+2m+2}\left(D\right)$, the function $f\left(t,x\right)$ satisfy the condition*

$$\left.\frac{\partial^i f\left(t,x\right)}{\partial t^{i_1}\partial x^{i_2}}\right|_{t=x} = 0, \;\; i = i_1 + i_2; \;\; i = 0, 1, ..., 2n + 2m + 2. \tag{2.18}$$

Then problem (2.17) has a unique solution, $W_0\left(t,x\right) \in C^{2n+2m+2}\left(D\right)$ and

$$\left.\frac{\partial^i W_0\left(t,x\right)}{\partial t^{i_1}\partial x^{i_2}}\right|_{t=x} = 0, \;\; i = i_1 + i_2; \;\; i = 0, 1, ..., 2n + 2m + 2. \tag{2.19}$$

Proof. The characteristical line of the equation in problem (2.17) passing through the origin of coordinates, divides the rectangle D into two parts $D_1 = \{(t,x) \in D|\, x \leq t\}$ and $D_2 = \{(t,x) \in D|\, x \geq t\}$. The solution of problem (2.17) may be sought in the form

$$W_0\left(t,x\right) = \begin{cases} W_0^{(1)}\left(t,x\right) & for \quad (t,x) \in D_1, \\ W_0^{(2)}\left(t,x\right) & for \quad (t,x) \in D_2, \end{cases} \tag{2.20}$$

the functions $W_0^{(1)}$ and $W_0^{(2)}$ are the solutions of the following boundary value problem

$$\frac{\partial W_0^{(1)}}{\partial t} + \frac{\partial W_0^{(1)}}{\partial x} + a\left(t,x\right)W_0^{(1)} = f\left(t,x\right); \;\; (t,x) \in D_1;$$

$$W_0^{(1)}\Big|_{x=0} = 0, \;\; (0 \leq t \leq 1), \tag{2.21}$$

$$\frac{\partial W_0^{(2)}}{\partial t} + \frac{\partial W_0^{(2)}}{\partial x} + a\left(t,x\right)W_0^{(2)} = f\left(t,x\right); \;\; (t,x) \in D_2;$$

$$W_0^{(2)}\Big|_{t=0} = 0, \;\; (0 \leq x \leq 1). \tag{2.22}$$

The solutions of problem (2.21), (2.22) are represented by the following formula:

$$W_0^{(1)}\left(t,x\right) = \int_0^x f\left(t - x + \tau, \tau\right)\exp\left[-\int_\tau^x a\left(t - x + \xi, \xi\right)d\xi\right]d\tau, \tag{2.23}$$

$$W_0^{(2)}\left(t,x\right) = \int_0^t f\left(\tau, \tau + x - t\right)\exp\left[-\int_\tau^t a\left(\xi, \xi + x - t\right)d\xi\right]d\tau. \tag{2.24}$$

Obviously, if $a\left(t,x\right)$ and $f\left(t,x\right)$ are rather smooth functions in D, then the functions $W_0^{(1)}\left(t,x\right)$ and $W_0^{(2)}\left(t,x\right)$ determined by formula (2.23), (2.24) are also smooth functions in D_1 and D_2 respectively. Therefore, smoothness of the function $W_0\left(t,x\right)$ represented by formula (2.20) in D for $x \neq t$ doesn't give rise to doubts. But the values of the functions $W_0^{(1)}(t,x)$ and $W_0^{(2)}(t,x)$ do not coincide on the line $x=t$. Consequently, the function $W_0(t,x)$ have breaks for $x=t$.

Using the explicit form (2.23), (2.24) of functions $W_0^{(1)}(t,x)$ and $W_0^{(2)}(t,x)$ it is easy to see that when $f(t,x)$ satisfies condition (2.18), all derivatives $W_0^{(1)}(t,x)$ and $W_0^{(2)}(t,x)$ to the $(2n+2m+2)$-th order inclusively vanish for $t=x$. Therefore, from (2.20) it follows that

$$\left.\frac{\partial^i W_0(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{x-t=-0} = \left.\frac{\partial^i W_0^{(1)}(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{t=x} =$$

$$= \left.\frac{\partial^i W_0^{(2)}(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{t=x} = \left.\frac{\partial^i W_0(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{x-t=+0} = 0.$$

Consequently, the functions $\frac{\partial^i W_0(t,x)}{\partial t^{i_1}\partial x^{i_2}}$; $i=1,2,...,2n+2m+2$ are continuous and vanish for $t=x$, i.e. condition (2.19) is fulfilled. Lemma 2.1 is proved.

From (2.13) for $i=0$ it follows that before constructing the function W_1 it is necessary to determine V_0. The function V_0 is the boundary layer type solution of equation (2.8) satisfying boundary conditions (2.16_0). The characteristical equation corresponding to ordinary differential equation (2.8) has m roots with negative real parts. Denote them by $\lambda_1,\lambda_2,...,\lambda_m$. It should be noted that the amount of lost boundary conditions on S_1 also equals m. Therefore, problem (1.6)-(1.9) regularly degenerates on S_1. It is easy to show that the boundary layer type solution of problem (2.8), (2.16_0) is of the form

$$V_0(\xi,x)=$$

$$=-\frac{\partial W_0(0,x)}{\partial t}\left[C_{01}\exp(\lambda_1\xi)+C_{02}\exp(\lambda_2\xi)+...+C_{0m}\exp(\lambda_m\xi)\right], \tag{2.25}$$

where C_{0i} are the known numbers.

As the functions W_0, V_0 are known, then we can already determine the function W_1 from problem (2.4), (2.5), (2.13) for $i=1$. The solution of this problem may be sought in the form $W_1 = W_1^{(1)}+W_1^{(2)}$, where $W_1^{(1)}$ and $W_1^{(2)}$ are the solutions of the following boundary value problems:

$$\frac{\partial W_1^{(1)}}{\partial t}+\frac{\partial W_1^{(2)}}{\partial t}+a(t,x)W_1^{(1)}=\frac{\partial^2 W_0}{\partial x^2};\quad \left.W_1^{(1)}\right|_{t=0}=0,\quad \left.W_1^{(1)}\right|_{x=0}=0, \tag{2.26}$$

$$\frac{\partial W_1^{(2)}}{\partial t}+\frac{\partial W_1^{(2)}}{\partial t}+a(t,x)W_1^{(2)}=0,\quad \left.W_1^{(2)}\right|_{t=0}=\varphi_1(x),\quad \left.W_1^{(2)}\right|_{x=0}=0. \tag{2.27}$$

Here $\varphi_1(x)$ is determined by the formula

$$\varphi_1(x)=-V_0(0,x)=-\left(\sum_{i=1}^m C_{0i}\right)\frac{\partial W_0(0,x)}{\partial t}. \tag{2.28}$$

The first part of the equation for $W_1^{(1)}$ satisfies condition (2.18) in lemma 2.1 for $i=0,1,...,2n+2m$. Therefore, by the same lemma, problem (2.26) has a unique solution, $W_1^1(t,x)\in C^{2n+2m}(D)$ and the function $W_1^{(1)}(t,x)$ satisfies the condition

$$\left.\frac{\partial^i W_1^{(1)}(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{t=x}=0;\ \ i=i_1+i_2;\ \ i=0,1,...,2n+2m. \tag{2.29}$$

The solution of problem (2.27) is represented by the formula

$$W_1^{(2)}(t,x) = \begin{cases} 0 & for\ \ 0 \le x < t \le 1, \\ \varphi_1(t-x)\exp\left[-\int\limits_0^t a(\tau, x-t+\tau)\,d\tau\right] & for\ \ 0 \le t < x \le 1. \end{cases} \tag{2.30}$$

By lemma 2.1, from (2.28) it follows that $\varphi_1(x) \in C^{2n+2m+1}[0,1]$. Therefore, smoothness in D for $t \neq x$, of the function $W_1^{(2)}(t,x)$ determined by formula (2.30) is obvious. From (2.19) and (2.28) we get

$$\varphi_1^{(k)}(0) = 0; \quad k = 0,1,...,2n+2m+1. \tag{2.31}$$

Taking into account (2.31), the smoothness of the function $W_1^{(2)}(t,x)$ and vanishing of all derivatives for $t = x$ is obtained directly from (2.30). Hence and from (2.29) it follows that the function $W_1(t,x)$ being the sum of $W_1^{(1)}, W_1^{(2)}$ belongs to the space $C^{2n+2m}(D)$ and satisfies the condition

$$\left.\frac{\partial^i W_1(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{t=0} = 0; \ \ i = i_1+i_2; \ \ i = 0,1,...,2n+2m. \tag{2.32}$$

The remaining functions $W_2, W_3, ..., W_n$ entering into the right hand side of (2.1) are constructed by the reasonings similar to ones carried out for W_1. Therefore we will not stop on their construction.

By constructing the functions $V_0, V_1, ..., V_{n+m-1}$ the following statement is used.

Lemma 2.2. *The functions V_j being the boundary layer type solutions of equation (2.9), satisfying appropriate boundary conditions from $(2.16_1)-(2.16_{n+m-1})$ are determined by the formula*

$$V_j(\xi,x) = \sum_{i=1}^{m}\left[C_{j0}^{(i)}(x) + C_{j1}^{(i)}(x)\,\xi + ... + C_{jj}^{(i)}(x)\,\xi^j\right]\exp(\lambda,\xi),$$

$$j = 1,2,...,n+m-1, \tag{2.33}$$

and the coefficients $C_{js}^{(i)}(x)$ are uniformly expressed by the function

$$\frac{\partial^k W_r(0,x)}{\partial t^{k_1+1}\partial x^{k_2}}; \ \ k = k_1+k_2+1; \ \ r = 0,1,...,j; \ \ k_1 = 0,1,...,m-1; \ k_1+k_2+r = j. \tag{2.34}$$

Proof. At first determine the function V_1. The function V_1 is the boundary layer type solution of equation (2.9) for $j = 1$, satisfying boundary condition (2.16_1). Taking into account explicit form (2.25) of the function $V_0(\xi,x)$, we represent the right hand side of equation (2.9) for $j = 1$ in the form

$$h_1(\xi,x) = \theta(x)\left[\sum_{i=1}^{m} C_{0i}\exp(\lambda_i\xi)\right], \tag{2.35}$$

and $\theta(x)$ is determined by the formula

$$\theta(x) = \frac{\partial^2 W_0(0,x)}{\partial t\partial x} + a(0,x)\,\frac{\partial W_0(0,x)}{\partial t}. \tag{2.36}$$

Obviously, the function

$$V_1^{(1)}(\xi,x) = \theta(x)\,\xi\left[\sum_{i=1}^{m} b_{0i}\exp(\lambda_i\xi)\right] \tag{2.37}$$

is a boundary layer type particular solution of equation (2.9), where b_{0i} are known numbers. Represent V_1 in the form $V_1 = V_1^{(1)} + V_1^{(2)}$. Then $V_1^{(2)}$ will be a boundary layer type solution of the following boundary value problem

$$(-1)^m \frac{\partial^{2m+1} V_1^{(2)}}{\partial \xi^{2m+1}} + \frac{\partial V_1^{(2)}}{\partial \xi} = 0, \tag{2.38}$$

$$\left.\frac{\partial V_1}{\partial \xi}\right|_{\xi=0} = d_1(x), \quad \left.\frac{\partial^2 V_1^{(2)}}{\partial \xi^2}\right|_{\xi=0} = d_2(x), ..., \left.\frac{\partial^m V_1^{(2)}}{\partial \xi^m}\right|_{\xi=0} = d_m(x). \tag{2.39}$$

Here the following denotation are used:

$$d_1(x) = -\left.\frac{\partial W_1}{\partial t}\right|_{t=0} + p_1\theta(x), \quad d_2(x) = -\left.\frac{\partial^2 W_1}{\partial t^2}\right|_{t=0} + p_2\theta(x),$$

$$d_j(x) = p_j\theta(x); \ \ j = 3, 4, ..., m;$$

$$p_s = -s \sum_{i=1}^{m} b_{0i}\lambda_i^{s-1}; \ \ s = 1, 2, ..., m.$$

It is easy to see that the boundary layer type solution of problem (2.38), (2.39) is of the form:

$$V_1^{(2)}(\xi, x) = C_1(x)\exp(\lambda_1\xi) + C_2(x)\exp(\lambda_2\xi) + ... + C_m(x)\exp(\lambda_m\xi). \tag{2.40}$$

Here the functions $C_i(x)$ are represented by the following formula:

$$C_i(x) = C_i^{(1)}\frac{\partial W_1(0,x)}{\partial t} + C_i^{(2)}\frac{\partial^2 W_0(0,x)}{\partial t^2} + C_i^{(3)}\frac{\partial^2 W_0(0,x)}{\partial t \partial x} +$$
$$+ C_i^{(4)} a(0,x)\frac{\partial W_0(0,x)}{\partial t}, \tag{2.41}$$

where $C_i^{(1)},\ C_i^{(2)},\ C_i^{(3)},\ C_i^{(4)};\ i = 1, 2, ..., m$ are known numbers.

From (2.37) and (2.40) we get that the function V_1 being the sum of $V_1^{(1)}$ and $V_1^{(2)}$ is determined by the formula:

$$V_1(\xi, x) = \sum_{i=1}^{n} [C_i(x) + b_{0i}\theta(x)\xi]\exp(\lambda_i\xi). \tag{2.42}$$

Introducing the denotation

$$C_{10}^{(i)}(x) = C_i(x), \ C_{11}^{(i)}(x) = b_{0i}\theta(x); \ \ i = 1, 2, ..., m, \tag{2.43}$$

we can write formula (2.42) as

$$V_1(\xi, x) = \sum_{i=1}^{m} \left[C_{10}^{(i)}(x) + C_{11}^{(i)}(x)\xi\right]\exp(\lambda_i\xi). \tag{2.44}$$

From (2.36), (2.41), (2.43), (2.44) it follows that the statement of lemma 2.2 is valid for $j = 1$.

Assuming that the statement of lemma 2.2 is valid for $j \leq r-1$, $(r < n + m - 1)$, and repeating the reasonings carried out when determining the function V_1, we can prove that it is valid for $j = r$ as well.

Lemma 2.2 is proved.

Multiply all V_j, by smoothing functions and denote the obtained new functions again by V_j; $j = 0, 1, ..., , n + m + 1$.

It is known that the functions $W_i\left(t, x\right);\ i = 0, 1, ..., n$ together will all their derivatives vanish for $x = t$ and in particular for $x = t = 0$. Consequently, from (2.25), (2.33), (2.34) we get that all functions $V_j\left(\xi, x\right)$; $j = 0, 1, ..., n + m - 1$ vanish for $x = 0$. Hence and from (2.1), (2.5), (2.11) it follows that the sum $W + V$ in addition to (2.10), (2.14) satisfies the boundary condition

$$\left.\left(W + V\right)\right|_{x=0} = 0 \tag{2.45}$$

as well. The constructed sum $W + V$ generally speaking, doesn't satisfy boundary conditions (1.9) on S_2. In this connection, it is necessary to construct a boundary layer type function near the boundary S_2.

Construction of boundary layer functions near the boundary S_2 is performed similar to construction of boundary layer functions near the boundary S_1. Therefore, we will not stop in detail on construction of boundary layer functions near the boundary S_2. We note only some moments.

Here, change of variables is made by the formulas: $1 - t = \varepsilon y$, $x = x$. The boundary layer type function η near the boundary S_2 should be sought in the form:

$$\eta = \varepsilon^{m+1}\left(\eta_0 + \varepsilon\eta_1 + ... + \varepsilon^{n+m-1}\eta_{n+m-1}\right), \tag{2.46}$$

as the solution of the equation

$$L_{\varepsilon,2}\left(W + V + \eta\right) - L_{\varepsilon,2}\left(W + V\right) = O\left(\varepsilon^{n+2m+1}\right), \tag{2.47}$$

where $L_{\varepsilon,2}$ is a new decomposition of the operator L_ε near the boundary S_2 in coordinates (y, x).

The equations for $\eta_0, \eta_1, ..., \eta_{n+m-1}$ have the same form as the equations for $V_0, V_1, ..., V_{n+m-1}$. The right hand sides of equations for $\eta_1, \eta_2, ..., \eta_{n+m-1}$ differ from the right hand sides of appropriate equations for $V_1, V_2, ..., V_{n+m-1}$ only by opposite signs.

The boundary conditions for equations whose solutions will be the functions $\eta_0, \eta_1, ..., \eta_{n+m-1}$ are found from the requirement that the sum $W + V + \eta$ should satisfy the following boundary conditions:

$$\left.\frac{\partial^{m+1}}{\partial t^{m+1}}\left(W + V + \eta\right)\right|_{t=1} = \\ = \left.\frac{\partial^{m+2}}{\partial t^{m+2}}\left(W + V + \eta\right)\right|_{t=1} = ... = \left.\frac{\partial^{2m}}{\partial t^{2m}}\left(W + V + \eta\right)\right|_{t=1} = 0. \tag{2.48}$$

The following statement is proved similar to the proof of lemma 2.2.

Lemma 2.3. *The boundary layer type functions near the boundary S_2 are determined by the formula*

$$\eta_j\left(y, x\right) = \sum_{i=1}^{m}\left[d_{j0}^{(i)}\left(x\right) + d_{j1}^{(i)}\left(x\right)y + ... + d_{jj}^{(i)}\left(x\right)y^j\right]\exp\left(\lambda_i y\right); \\ j = 0, 1, ..., n + m - 1. \tag{2.49}$$

Here the coefficients $d_{js}^{(i)}\left(x\right)$ are expressed by the function

$$\frac{\partial^{m+1+k}W_r\left(1, x\right)}{\partial t^{m+1+k_1}\partial x^{k_2}}; \\ k = k_1 + k_2;\ k_1 = 0, 1, ..., m - 1;\ r = 0, 1, ..., j;\ k_1 + k_2 + r = j. \tag{2.50}$$

Multiply all functions η_j by the smoothing function and for the obtained new functions leave the previous denotation η_j; $j = 0, 1, ..., n + m - 1$.

As the function η vanishes at the expense of the smoothing function for $t = 0$, then from (2.10) and (2.14) if follows that the sum $W + V + \eta$ along with conditions (2.48) satisfies the following boundary conditions as well:

$$(W + V + \eta)|_{t=0} = \varepsilon^{n+1}\varphi_\varepsilon(x), \ \frac{\partial}{\partial t}(W + V + \eta)\bigg|_{t=0} =$$

$$= \frac{\partial^2}{\partial t^2}(W + V + \eta)\bigg|_{t=0} = 0, ..., \frac{\partial^m}{\partial t^m}(W + V + \eta)\bigg|_{t=0} = 0, \tag{2.51}$$

where $\varphi_\varepsilon(x)$ is determined by formula (2.15).

Following (2.45) and (2.46), we have that if all functions η_j vanish for $x = 0$, i.e.

$$\eta_j|_{x=0} = 0; \ \ j = 0, 1, ..., n + m - 1, \tag{2.52}$$

then the sum $W + V + \eta$ in addition to (2.48), (2.51) will satisfy the boundary condition

$$(W + V + \eta)|_{x=0} = 0, \tag{2.53}$$

as well.

If follows from (2.49), (2.50) that for fulfilment of conditions (2.52) it is suffices that the functions W_i satisfy the conditions

$$\frac{\partial^{m+1+k}W_i(1,0)}{\partial t^{m+1+k_1}\partial x^{k_2}} = 0;$$

$$k = k_1 + k_2; \ \ i = 0, 1, ..., n; \ \ k_1 + k_2 + r = 0, 1, ..., n + m - 1. \tag{2.54}$$

Assume that the function $f(t, x)$ satisfies the condition

$$\frac{\partial^k f(1,0)}{\partial t^{k_1}\partial x^{k_2}} = 0; \ \ k = k_1 + k_2; \ \ k = 0, 1, ..., n + m - 1. \tag{2.55}$$

Then conditions (2.54) and consequently (2.52), (2.53) will be satisfied.

Thus, the constructed sum $W + V + \eta$ satisfies boundary conditions (2.48), (2.51), (2.53). But this sum do not satisfy boundary conditions of the second boundary condition from (1.9) for $x = 1$. Therefore we should construct a boundary layer type function ψ near the boundary S_3 so that ψ could ensure fulfilment of the boundary condition

$$(W + V + \eta + \psi)|_{x=1} = 0. \tag{2.56}$$

By constructing the function ψ we should care that the equality

$$L_{\varepsilon,3}(W + V + \eta + \psi) - L_{\varepsilon,3}(W + V + \eta) = O\left(\varepsilon^{n+1}\right) \tag{2.57}$$

be fulfilled. Here $L_{\varepsilon,3}$ denotes a new decomposition of the operator L_ε near the boundary S_3 in coordinates (t, τ), where $\tau = \frac{1-x}{\varepsilon}$. We should look for the boundary layer function ψ in the form

$$\psi = \psi_0(t, \tau) + \varepsilon\psi_1(t, \tau) + ... + \varepsilon^{n+1}\psi_{n+1}(t, \tau). \tag{2.58}$$

Having substituted expression (2.58) of the function ψ and new expansions in a small parameter of already constructed functions W, V, η in coordinates (t, τ) in (2.57) we get the following equations whose solutions are the functions $\psi_0, \psi_1, ..., \psi_{n+1}$:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \psi_0}{\partial \tau}\right)^p + \frac{\partial^2 \psi_0}{\partial \tau^2} + \frac{\partial \psi_0}{\partial \tau} = 0, \tag{2.59}$$

$$\frac{\partial}{\partial \tau}\left\{\left[p\left(\frac{\partial \psi_0}{\partial \tau}\right)^{p-1} + 1\right]\frac{\partial \psi_j}{\partial \tau}\right\} + \frac{\partial \psi_j}{\partial \tau} = \Phi_j;\ j = 1, 2, ..., n+1. \tag{2.60}$$

Here Φ_j are known functions polynomially depending on the first and second derivatives of the functions $V_0, V_1, ..., V_{j-1}$.

We now find boundary conditions for equations (2.59), (2.60). As is known, all functions $W_i(t, x)$; $i = 0, 1, ..., n$ vanish for $t = x$, in particular for $t = x = 1$. Taking this fact into account, from (2.49) and (2.50) we get that all functions $\eta_j;\ j = 0, 1, ..., n+m-1$ vanish for $x = 1$, whence we have $\eta|_{x=1} = 0$. Hence it follows that we can write equality (2.56) in the form

$$\left.(W + V + \psi)\right|_{x=1} = 0 \tag{2.61}$$

Assume that the function $f(t, x)$ satisfies the condition

$$\frac{\partial^k f(0,1)}{\partial t^{k_1} \partial x^{k_2}} = 0;\ k = k_1 + k_2;\ \ k = 0, 1, ..., n+m-1. \tag{2.62}$$

Then from (2.33), (2.34) it follows that all functions $V_j;\ j = 0, 1, ..., n+m-1$ will vanish for $x = 1$, hence we have $V|_{x=1} = 0$. Therefore, equality (2.61) takes the form

$$\left.(W + \psi)\right|_{x=1} = 0. \tag{2.63}$$

Having substituted expressions (2.1), (2.58) for W, ψ in (2.63), and taking into account the fact that all functions $\psi_j;\ j = 0, 1, ..., n+1$, should be boundary layer type functions from (2.63) we get the following conditions for equations (2.59), (2.60):

$$\psi_j|_{\tau=0} = \varphi_j(t)\ ,\quad \lim_{\tau \to +\infty} \psi_j = 0, \tag{2.64}$$

where $\varphi_j(t) = -W_j(t, 1)$, for $j = 0, 1, ..., n$ and $\varphi_{n+1} \equiv 0$.

The following statement is valid.

Lemma 2.4. *For every $t \in [0,1]$ problem (2.59), (2.64) for $j = 0$ has a unique solution, and the function $\psi_0(t, \tau)$ to τ with respect to τ is continuously differentiable with respect to t, has continuous derivatives to $2(n+m+1)$-th order, inclusively. The following estimation is valid:*

$$\left|\frac{\partial^i \psi_0(t,\tau)}{\partial t^{i_1} \partial \tau^{i_2}}\right| \le G_{i_1 i_2}\left(|\varphi_0(t)|, \left|\varphi_0^{'}(t)\right|, ..., \left|\varphi_0^{(i_1)}(t)\right|\right)\exp(-\tau), \tag{2.65}$$

where $i = i_1 + i_2$; $i = 0, 1, ..., 2n+m+2$; $G_{i_1 i_2}(t_1, t_2, ..., t_{i_1+1})$ are some known polynomials of their own arguments with non-negative coefficients, free terms of these polynomials equal zero, and at least one of other coefficients is non zero.

The proof of lemma 2.4 is in [34, theorem 2], (see also [39, theorem 2]).

Construction of remaining functions $\psi_1, \psi_2, ..., \psi_{n+1}$ as the solutions of linear problems (2.60), (2.64) for $j = 1, 2, ..., n+1$ is based on the following statement, whose prove is given in [34, Theorem 3].

Lemma 2.5. *Problems (2.60), (2.64) for $j = 1, 2, ..., n+1$ have unique solutions, the functions $\psi_j(t,\tau)$; $j = 1, 2, ..., n+1$ have continuous derivatives to the $2(n+m+1-j)$-th order, inclusively. The following estimations are valid:*

$$\left|\frac{\partial^i \psi_j(t,\tau)}{\partial t^{i_1} \partial \tau^{i_2}}\right| \leq \sum_{s=0}^{i_1+j+1} |a_{js}(t)| \tau^s \exp(-\tau), \tag{2.66}$$

where $i = i_1 + i_2$; $i = 0, 1, ..., 2n + 2m + 2 - j$; $a_{js}(t)$ are known functions expressed by the functions $\varphi_0(t)$, $\varphi_1(t), ..., \varphi_j(t)$ and their derivatives in the form of polynomials without free terms $j = 1, 2, ..., n+1$.

Multiply all functions $\psi_j; j = 0, 1, ..., n+1$ by the smoothing cofactor and for the obtained new functions leave the previous denotation.

So, we constructed the sum

$$\tilde{u} = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{s=0}^{n+m-1} \varepsilon^{1+s} V_s + \sum_{s=0}^{n+m-1} \varepsilon^{1+m+s} \eta_s + \sum_{j=0}^{n+1} \varepsilon^i \psi_j, \tag{2.67}$$

that following (2.56) satisfies the boundary condition

$$\tilde{u}|_{x=1} = 0. \tag{2.68}$$

As the function ψ vanishes for $x = 0$ at the expense of the smoothing cofactor, it follows from (2.53) that in addition to condition (2.68), $\tilde{u}$ satisfies the boundary condition

$$\tilde{u}|_{x=0} = 0, \tag{2.69}$$

as well.

From (2.62) it follows that the functions $\varphi_j(t) = -W_j(t,1)$; $j = 0, 1, ..., n$ satisfy the following conditions

$$\varphi_j^{(k)}(0) = 0; \; k = 0, 1, ..., m; \; j = 0, 1, ..., n. \tag{2.70}$$

Then from estimations (2.65), (2.66) and from (2.70) we have

$$\left.\frac{\partial^k \psi_j(t,\tau)}{\partial t^k}\right|_{t=0} = 0; \quad k = 0, 1, ..., m; \; j = 0, 1, ..., n+1. \tag{2.71}$$

From (2.58) and (2.71) we get

$$\psi|_{t=0} = \left.\frac{\partial \psi}{\partial t}\right|_{t=0} = ... = \left.\frac{\partial^m \psi}{\partial t^m}\right|_{t=0} = 0. \tag{2.72}$$

Taking into account (2.51), (2.72) we have that the sum $\tilde{u} = W + V + \eta + \varphi$ alongside with conditions (2.68), (2.69) satisfies the following boundary conditions as well

$$\tilde{u}|_{t=0} = \varepsilon^{n+1} \varphi_\varepsilon(x), \; \left.\frac{\partial \tilde{u}}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 \tilde{u}}{\partial t^2}\right|_{t=0} = ... = \left.\frac{\partial^m \tilde{u}}{\partial t^m}\right|_{t=0} = 0. \tag{2.73}$$

It is known that all functions $W_i(t,x)$; $i = 0, 1, ..., n$ together with their own derivatives vanish at $t = x$, in particular for $t = x = 1$. Hence it follows that the function $\varphi_j(t) = -W_j(t,1)$ satisfies the conditions:

$$\varphi_j^{(k)}(1) = 0; \; k = 0, 1, ..., 2m. \tag{2.74}$$

From estimations (2.65), (2.66) and (2.74) we get that

$$\left.\frac{\partial^{m+k}\psi_j(t,\tau)}{\partial t^{m+1}}\right|_{i=1} = 0; \quad k = 1,2,...,m; \;\; j = 0,1,...,n+1. \tag{2.75}$$

From (2.58) and (2.75) it follows that

$$\left.\frac{\partial^{m+1}\psi}{\partial t^{m+1}}\right|_{t=1} = \left.\frac{\partial^{m+2}\psi}{\partial t^{m+2}}\right|_{t=1} = ... = \left.\frac{\partial^{2m}\psi}{\partial t^{2m}}\right|_{t=1} = 0. \tag{2.76}$$

Following (2.48), (2.76) we get that the function $\tilde{u}$ in addition to conditions (2.68), (2.69), (2.73), satisfies the following boundary conditions as well:

$$\left.\frac{\partial^{m+1}\tilde{u}}{\partial t^{m+1}}\right|_{t=1} = \left.\frac{\partial^{m+2}\tilde{u}}{\partial t^{m+2}}\right|_{t=1} = ... = \left.\frac{\partial^{2m}\tilde{u}}{\partial t^{2m}}\right|_{t=1} = 0. \tag{2.77}$$

Introduce the denotation

$$u - \tilde{u} = z \tag{2.78}$$

and call the function z a remainder term, where u is the solution of problem (1.6)-(1.9). Then (2.67), (2.78) yields the following asymptotic expansion in a small parameter of the solution of problem (1.6)-(1.9):

$$u = \sum_{i=0}^{n}\varepsilon^i W_i + \sum_{s=0}^{n+m-1}\varepsilon^{1+s}V_s + \sum_{s=0}^{n+m-1}\varepsilon^{1+m+s}\eta_s + \sum_{j=0}^{n+1}\varepsilon^j\psi_j + z. \tag{2.79}$$

Now we should estimate the remainder term.

3 Estimating the Remainder Term, and Formulation of the Main Result

The following lemma is valid.

Lemma 3.1. For the remainder term z in (2.79) the following estimation is valid

$$\varepsilon^{2m}\int\limits_0^1\left(\left.\frac{\partial^m z}{\partial t^m}\right|_{t=1}\right)^2 dx + \varepsilon^p\iint\limits_D\left(\frac{\partial z}{\partial x}\right)^{p+1} dtdx + \varepsilon\iint\limits_D\left(\frac{\partial z}{\partial x}\right)^2 dtdx +$$

$$+ C_1\iint\limits_D z^2 dtdx \leq C_2\varepsilon^{2(n+1)}, \tag{3.1}$$

where $C_1 > 0$, $C_2 > 0$ are constants independent of ε.

Proof. Putting together (2.2), (2.7), (2.47), (2.57), we have that the function $\tilde{u}$ satisfies the equation

$$L_\varepsilon\tilde{u} = O\left(\varepsilon^{n+1}\right). \tag{3.2}$$

Subtracting (3.2) from (1.6), we get

$$(-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} z}{\partial t^{2m+1}} - \varepsilon^p \frac{\partial}{\partial x} \left[\left(\frac{\partial u}{\partial x} \right)^p - \left(\frac{\partial \tilde{u}}{\partial x} \right)^p \right] - \varepsilon^2 \frac{\partial^2 z}{\partial x^2} +$$

$$+ \frac{\partial z}{\partial t} + az = O\left(\varepsilon^{n+1}\right). \tag{3.3}$$

From (1.7)-(1.9), (2.68), (2.69), (2.73), (2.77) it follows that z satisfies the following boundary conditions

$$z|_{t=0} = -\varepsilon^{n+1} \varphi_\varepsilon(x), \quad \left. \frac{\partial^k z}{\partial t^k} \right|_{t=0} = 0; \quad \left. \frac{\partial^{m+k} z}{\partial t^{m+k}} \right|_{t=1} = 0; \;\; k = 1, 2, ..., m, \tag{3.4}$$

$$z|_{x=0} = z|_{x=1} = 0. \tag{3.5}$$

Here the function $\varphi_\varepsilon(x)$ is determined by formula (2.15) and satisfies the conditions

$$\varphi_\varepsilon(0) = \varphi_\varepsilon(1) = 0. \tag{3.6}$$

Let us consider the auxiliary function

$$z_1 = \varepsilon^{n+1} \left[t^{m+1} (1-t)^{2m+1} x (1-x) - \varphi_\varepsilon(x) \right]. \tag{3.7}$$

It is easy to see that the function z_1 determined by formula (3.7), satisfies boundary conditions (3.4), (3.5) as well.

Represent the remainder term z in the form

$$z = z_1 + z_2. \tag{3.8}$$

Obviously, the function z_2 satisfies the following boundary conditions

$$\left. \frac{\partial^k z_2}{\partial t^k} \right|_{t=0} = 0; \;\; k = 0, 1, ..., m; \quad \left. \frac{\partial^{m+k} z_2}{\partial t^{m+k}} \right|_{t=0} = 0; \;\; k = 1, 2, ..., m, \tag{3.9}$$

$$z_2|_{x=0} = z_2|_{x=1} = 0. \tag{3.10}$$

Having substituted the expression of z from (3.8) in (3.3) and taking into account (3.7), after some transformations we get the equation

$$(-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} z_2}{\partial t^{2m+1}} - \varepsilon^p \frac{\partial}{\partial x} \left\{ \left[\frac{\partial (z_2 + \tilde{u} + z_1)}{\partial x} \right]^p - \left[\frac{\partial (\tilde{u} + z_1)}{\partial x} \right]^p \right\} -$$

$$- \varepsilon^p \frac{\partial}{\partial x} \left\{ \left[\frac{\partial (\tilde{u} + z_1)}{\partial x} \right]^p - \left(\frac{\partial \tilde{u}}{\partial x} \right)^p \right\} - \varepsilon \frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_2}{\partial x} + \frac{\partial z_2}{\partial t} + a z_2 = O\left(\varepsilon^{n+1}\right). \tag{3.11}$$

Taking into account (3.7), we can show that the term $\frac{\partial}{\partial x} \left\{ \left[\frac{\partial (\tilde{u}+z_1)}{\partial x} \right]^p - \left(\frac{\partial \tilde{u}}{\partial x} \right)^p \right\}$ has order of smallness $O\left(\varepsilon^{n+1}\right)$ with respect to ε. Taking this term to the right hand side of (3.11) we get the following equation

$$(-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} z_2}{\partial t^{2m+1}} - \varepsilon^p \frac{\partial}{\partial x} \left\{ \left[\frac{\partial (z_2 + \tilde{u} + z_1)}{\partial x} \right]^p - \left[\frac{\partial (\tilde{u} + z_1)}{\partial x} \right]^p \right\} -$$

$$- \varepsilon \frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_2}{\partial x} + \frac{\partial z_2}{\partial t} + a z_2 = O\left(\varepsilon^{n+1}\right). \qquad (3.12)$$

Multiplying both hand sides of (3.12) by z_2, integrating by parts the obtained expressions on domain D, allowing for conditions (3.9), (3.10) after certain transformations we get the validity of estimation (3.1) for the function z_2. From (3.7), (3.8) and from the estimation for z_2 we get validity of estimation (3.1) for z.

Lemma 3.1 is proved.

4 Conclusion

Summarizing the obtained results, we arrive at the following statement.

Theorem 4.1. *Assume that the function $f(t,x) \in C^{2n+2m+2}(D)$ and satisfies conditions (2.18), (2.55), (2.62). Then for the solution of boundary value problem (1.6)-(1.9) the asymptotic representation (2.79) is valid, where the functions W_j are determined by the first iterative process, V_s, η_s, ψ_j are boundary layer type functions near the boundaries $t = 0$, $t = 1$ and $x = 1$ and also are determined by appropriate iterative processes, z is a remainder term and estimation (3.1) is valid for it.*

Competing Interests

Authors have declared that no competing interests exist.

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