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Hamiltonian Structure, Soliton Solution and Conservation Laws for a New Fifth-Order Nonlinear Evolution Equation Which Describes Pseudo-Spherical Surfaces

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Abstract

In this paper, we shall show that the Hamiltonian structure can be defined for any nonlinear evolution equations which describe surfaces of a constant negative curvature, so that the densities of conservation laws can be considered as corresponding Hamiltonians. This paper obtains the soliton solution and conserved quantities of a new fifth-order nonlinear evolution equation by the aid of inverse scattering method.

Keywords

Nonlinear Evolution Equations, Conservation Laws, Pseudo-Spherical Surfaces

1. Introduction

The notion of pseudo-spherical surfaces (pss) (surfaces of a constant negative curvature $k \equiv -1$) appeared in geometry in the middle of the nineteenth century. It was an important step in the development of mathematics. Pss become the final factor in the visual interpretation of non-Euclidean hyperbolic geometry discovered by Klingenberg [1]. The further development of mathematics found a close connection between pss and theory of nets, theory of solitons, attractors, some nonlinear evolution equations (NLEEs) of mathematical physics, Bäcklund transformations (BTs), and so on [2] [3] [4] [5]. The connection between geometry and the nonlinear partial differential equations (NLPDEs) has been studied in mathematical physics for more than a century. For instance, the classical Liouville equation describes minimal surfaces in the space E^3, and the sine Gordon equation is related to the geometry of pss, *i.e.*, surfaces with a negative Gaussian curvature [6] [7] [8] [9].

Conservation law plays a vital role in the study of nonlinear evolution equations, particularly with regard to integrability, linearization and constants of motion. In the present paper, it is shown that infinitely many conservation laws for certain nonlinear evolution equations are systematically constructed with symbolic computation in a simple way from the Riccati form of the Lax pair. Noting that the Lax pairs investigated here are associated with different linear systems, including the generalized Kaup-Newell (KN) spectral problem, the generalized Ablowitz-Kaup-Newell-Segur (AKNS) spectral problem, the generalized AKNS-KN spectral problem and a recently proposed integrable system. Therefore, the power and efficiency of this systematic method are well understood, and we expect it may be useful for other nonlinear evolution models, even higher-order and variable-coefficient ones [10]-[15].

Hamiltonians are of great importance in their own right and have found a remarkable number of applications in both physics and mathematics. Hamiltonians play a central role in the field of integrable systems and also play a fundamental role in several others areas of mathematics and physics. Hamiltonians are often referred to as the master integrable system. Hamiltonians provide as with a means of generating and classifying many integrable systems and they also give a unified geometrical framework in which to analyze them. Moreover, in the context of the inverse scattering transform, an integrable equation admits well-behaved solutions obtained via the related linear problems [16]-[22].

The main aim of this paper is to use the BTs in the construction of exact soliton solutions for a new fifth-order nonlinear evolution equation which describes pss. An infinite number of conservation laws are derived for a new fifth-order nonlinear evolution equation just mentioned using the corresponding Hamiltonians.

The latter yields directly the curvature condition (Gaussian curvature equal to -1, corresponding to pseudo-spherical surfaces). This geometrical method allows some further generalizations of the work on conservation laws given by Khater *et al.* [23]. An infinite number of conservation laws for a new fifth-order nonlinear evolution equation are derived in this way.

The paper is organized as follows. In Section 2, we introduce the inverse scattering method and apply the geometrical method to obtain Hamiltonian structure for any nonlinear evolution equations which describe surfaces of a constant negative curvature. In Section 3, a new exact soliton solution and the corresponding Hamiltonians are obtained for a new fifth-order nonlinear evolution equation. Section 4 contains the conclusion.

2. Hamiltonian Structure

The inverse scattering transform method allows one to linearize a large class of nonlinear evolution equations and can be considered as a nonlinear version of the Fourier transform [24] [25] [26] [27] [28]. An essential prerequisite of inverse scattering transform method is the association of the nonlinear evolution equation with a pair of linear problems (Lax pair), a linear eigenvalue problem, and a second associated linear problem, such that the given equation results as a compatibility condition between them [29] [30] [31] [32] [33]. Consider the following AKNS eigenvalues problem:

$$\psi_x = P\psi, \quad \psi_t = Q\psi, \tag{1}$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, P and Q are two 2×2 null-trace matrices

$$P = \begin{pmatrix} \frac{\eta}{2} & q \\ r & -\frac{\eta}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{2}$$

and η is a parameter independent of x and t, while q and r are assumed to be functions of x and t. From Equations (1) and (2), we get the following scattering problem:

$$\begin{aligned} \psi_{1x} &= \frac{\eta}{2}\psi_1 + q\psi_2, \\ \psi_{2x} &= r\psi_1 - \frac{\eta}{2}\psi_2, \end{aligned} \tag{3}$$

in which eignfunctions ψ_1 and ψ_2 evolve in time according to

$$\begin{aligned} \psi_{1t} &= A(x,t;\eta)\psi_1 + B(x,t;\eta)\psi_2, \\ \psi_{2t} &= C(x,t;\eta)\psi_1 - A(x,t;\eta)\psi_2. \end{aligned} \tag{4}$$

The integrability conditions reads

$$P_t - Q_x + [P,Q] = 0, \tag{5}$$

or in component form

$$\begin{aligned} &A_x = qC - rB, \\ &q_t - 2Aq - B_x + \eta B = 0, \\ &C_x = r_t + 2Ar - \eta C, \end{aligned} \tag{6}$$

Konno and Wadati [34] introduced the function

$$\Gamma = \frac{\psi_1}{\psi_2}. \tag{7}$$

Differentiating Equation (7) with respect to x and t, respectively, and using Equations (3), (4) and (7), then Equation (1) are reduced to the Riccati equations:

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma - r\Gamma^2 + q, \quad \frac{\partial \Gamma}{\partial t} = 2A\Gamma - C\Gamma^2 + B. \tag{8}$$

Now we construct a transformation Γ' satisfies the potential and then deduce a BTs for the considered nonlinear evolution equation

$$u' = u_0 + f(\Gamma, \eta), \tag{9}$$

where u_0 is the old solution and u' is a new solution corresponding nonlinear evolution equation. In order to determine conserved densities and fluxes, we can written Equation (8) in the form

$$C\frac{\partial\Gamma}{\partial x}-r\frac{\partial\Gamma}{\partial t}=\left(qC-Br\right)+\left(\eta C-2Ar\right)\Gamma. \tag{10}$$

Adding $-r_t\Gamma$ to both sides and using Equation (6), then Equation (10) takes the form of conservation laws

$$\frac{\partial\left(r\Gamma\right)}{\partial t}=\frac{\partial\left(-A+C\Gamma\right)}{\partial x}, \tag{11}$$

where $r\Gamma$ are conserved densities and $\left(-A+C\Gamma\right)$ are fluxes.

From Equation (3), I obtain

$$\frac{\psi_{2x}}{\psi_2}=-\frac{\eta}{2}+r\Gamma, \tag{12}$$

then

$$\ln\psi_2=-\left(\frac{\eta}{2}\right)x+\int r\Gamma\mathrm{d}x. \tag{13}$$

By rearranged the first equation of Equation (8) to take the form

$$\eta\left(r\Gamma\right)=-rq+\left(r\Gamma\right)^2-r\left(\frac{r\Gamma}{r}\right)_x, \tag{14}$$

then, I can expand $r\Gamma$ into a power series in inverse power of η as follows [28]

$$r\Gamma\left(x,t;\eta\right)=\sum_{n=1}^{\infty}\phi_n\left(x,t\right)\eta^{-n}. \tag{15}$$

By the same way, I expand $\ln\psi_2$ into a power series in inverse power of η so that

$$\ln\psi_2=H_0+\sum_{l=1}^{\infty}H_l\eta^{-l}, \tag{16}$$

where H_0 and H_l are Hamiltonians (conserved quantities), by substituting (15) into (14), the following system of conservation laws appears

$$\sum_{n=1}^{\infty}\phi_n\left(x,t\right)\eta^{-n+1}=-rq+\left(\sum_{n=1}^{\infty}\phi_n\left(x,t\right)\eta^{-n}\right)^2+r\left(\sum_{n=1}^{\infty}\left(\frac{\phi_n\left(x,t\right)}{r}\right)_x\eta^{-n}\right). \tag{17}$$

Now equate powers of η on both sides of this expression to produce the set of recursions,

$$\phi_1\left(x,t\right)=-qr,\quad \phi_2\left(x,t\right)=-rq_x,\quad \phi_{n+1}=\sum_{k=1}^{n-1}\phi_k\left(x,t\right)\phi_{n-k}\left(x,t\right)+r\left(\frac{\phi_n\left(x,t\right)}{r}\right)_x,\quad n\geq 2. \tag{18}$$

Then, by equate powers of η on both sides of Equations (13), (15) and (16), we obtain the infinite number of Hamiltonians may explicitly be determined in terms of smooth real functions $\phi_n\left(x,t\right)$ and their derivatives, as follows

$$H_0=-\left(\frac{\eta}{2}\right)x,\qquad H_1=\int\phi_1\mathrm{d}x=-\int qr\mathrm{d}x,$$

$$H_2 = \int \phi_2 \mathrm{d}x = -\int r q_x \mathrm{d}x,$$

Hamiltonians in general form

$$H_{n+1}(x,t) = \int \left(\sum_{k=1}^{n-1} \phi_k(x,t)\phi_{n-k}(x,t) + r\left(\frac{\phi_n(x,t)}{r}\right)_x \right) \mathrm{d}x, \quad n \geq 2. \tag{19}$$

The explicit expressions of the first few order Hamiltonians are

$$H_3 = \int \phi_3 dx = \int \left(q^2 r^2 - r q_{xx}\right) \mathrm{d}x, \qquad H_4 = \int \phi_4 \mathrm{d}x = \int \left(4r^2 q q_x + r q^2 r_x - r q_{3x}\right) \mathrm{d}x,$$

$$H_5 = \int \phi_5 \mathrm{d}x = \int \left[6r^2 q q_{2x} + 6 r q r_x q_x + 4 r^2 q_x^2 + r q^2 r_{2x} - 2 q^3 r^3 - r q_{4x}\right] \mathrm{d}x.$$

The procedure is clarified in the following example.

3. Soliton Solution and an Infinite Number of Conserved Quantities for a New Fifth-Order Nonlinear Evolution Equation

Now we consider a new fifth-order nonlinear evolution equation

$$u_t = u_{5x} - \frac{5}{2}u^2 u_{3x} - 10 u u_x u_{2x} - \frac{5}{2}u_x^3 + \frac{15}{8}u^4 u_x, \tag{20}$$

for $u(x,t)$ which describes a pss. There exist functions f_{ij}, $1 \leq i \leq 3$, $1 \leq j \leq 2$, which depend on $u(x,t)$ and its derivatives such that, for any solution u of the evolution equation, f_{ij} satisfy (5). For Equation (20) we consider the functions defined by [33]

$$f_{11} = u, \qquad f_{12} = u_{4x} + \left(\frac{-5}{2}u^2 - 3\eta^2\right)u_{2x} - \frac{5}{2}u u_x^2 + \frac{3}{8}u^5 + \frac{3}{2}\eta^2 u^3 + 9\eta^4 u,$$

$$f_{21} = \eta, \qquad f_{22} = -\eta\left(2u_{3x} - 3u^2 u_x - 6\eta^2 u_x + u u_{2x} - \frac{1}{2}u_x^2 - \frac{3}{8}u^4 - \frac{3}{2}\eta^2 u^2 - 9\eta^4\right), \tag{21}$$

$$f_{31} = 2\eta, \qquad f_{32} = -\eta\left(u_{3x} - 32u^2 u_x - 3\eta^2 u_x + 2u u_{2x} - u_x^2 - \frac{3}{4}u^4 - 3\eta^2 u^2 - 18\eta^4\right).$$

For any solution $u(x,t)$ of a new fifth-order evolution Equation (20), the matrices P and Q are

$$P = \begin{pmatrix} \frac{\eta}{2} & \frac{u}{2} - \eta \\ \frac{u}{2} + \eta & -\frac{\eta}{2} \end{pmatrix}, \tag{22}$$

$$Q = \begin{pmatrix} \frac{f_{22}}{2} & \frac{f_{12} - f_{32}}{2} \\ \frac{f_{12} + f_{32}}{2} & -\frac{f_{22}}{2} \end{pmatrix}, \tag{23}$$

where f_{22}, f_{12} and f_{32} defined by (21). The above matrices P, Q satisfy the Equation (5). Then the first equation of (8) becomes

$$\frac{\partial \Gamma}{\partial x} = \frac{\eta}{2}\left(\Gamma - \Gamma^2 - 1\right) + \frac{u}{2}\left(1 - \Gamma^2\right). \tag{24}$$

If we choose Γ' and u' as

$$\Gamma' = \frac{1}{\Gamma}, \tag{25}$$

$$u' = -u + 4\frac{\partial}{\partial x}\tanh^{-1}\Gamma, \tag{26}$$

then Γ' and u' satisfies Equation (24).

Now we shall choose some known solutions of the above a new fifth-order evolution equation and substitute these solutions into the corresponding matrices P and Q. Next, we solve Equations (3) for ψ_1 and ψ_2. Then, by (9) and the corresponding BT we shall obtain the new solutions for a new fifth - order evolution equation. I choose the known solution is a constant u_0, then substitute $u = u_0$ into the matrices P and Q in (22) and (23), then by (1) we have

$$\mathrm{d}\psi = \psi_x \mathrm{d}x + \psi_t \mathrm{d}t = P\psi \mathrm{d}\rho, \tag{27}$$

where

$$P = \begin{pmatrix} \frac{\eta}{2} & \frac{u_0}{2} - \eta \\ \frac{u_0}{2} + \eta & -\frac{\eta}{2} \end{pmatrix}, \tag{28}$$

$$\rho = x + \alpha t, \quad \alpha = \frac{3u_0^4}{8} + \frac{3\eta^2 u_0^2}{2} + 9\eta^4. \tag{29}$$

The solution of Equation (27) is

$$\psi = (\exp \rho P)\psi_0 = \left(I + \rho P + \frac{\rho^2 P^2}{2!} + \frac{\rho^3 P^3}{3!} + \cdots \right)\psi_0, \tag{30}$$

where ψ_0 is a constant column vector. The solution (30) takes the following form:

$$\psi = \begin{bmatrix} \cosh \alpha\rho + \frac{\eta}{2\alpha}\sinh \alpha\rho & \left(1 - \frac{\eta}{\alpha}\right)\sinh \alpha\rho \\ \left(1 + \frac{\eta}{\alpha}\right)\sinh \alpha\rho & \cosh \alpha\rho - \frac{\eta}{2\alpha}\sinh \alpha\rho \end{bmatrix}\psi_0. \tag{31}$$

Now, we choose $\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, in (31) and use (7) and the BT (26); we obtain the new solution class of the new fifth-order evolution Equation (20) corresponding to the known constant solution u_0 as follows

$$u = -u_0 - \frac{4a\alpha \operatorname{csch}^2 \alpha\rho}{1 - (b + a\coth \alpha\rho)^2}, \tag{32}$$

where $a = \frac{\alpha}{\alpha + \eta}, b = \frac{\eta}{2\alpha + 2\eta}$. If we choose $\psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in (31), we will obtain another solution. Obviously all of these solutions are traveling waves with velocity $\alpha = \frac{3u_0^4}{8} + \frac{3\eta^2 u_0^2}{2} + 9\eta^4$.

From Equations (19) and (22) the first few order Hamiltonians are determined by the relation

$$H_0 = -\left(\frac{\eta}{2}\right)x,\ H_1 = -\int\left(\frac{u^2}{4}-\eta^2\right)\mathrm{d}x,\ H_2 = -\int\frac{u_x}{2}\left(\frac{u}{2}+\eta\right)\mathrm{d}x,\ \text{etc.} \tag{33}$$

From the solution $u(x,t)$ of a new fifth-order evolution equation. This Hamiltonians (conserved quantities) given by the relation

$$\begin{aligned} H_0 &= -\left(\frac{\eta}{2}\right)x, \\ H_1 &= \left(\eta^2 - \frac{u_0^2}{4}\right)x + \frac{4\alpha^2\eta\sinh 2\alpha\rho}{(2\alpha+\eta)(-4\alpha-\eta+\eta\coth 2\alpha\rho)} \\ &\quad - \frac{2\sqrt{2}\sqrt{\alpha}\left(4\alpha^2-2u_0\alpha+\alpha\eta-u_0\eta\right)\operatorname{arctanh}\left(\sqrt{1+\frac{\eta}{2\alpha}}\tanh\alpha\rho\right)}{(2\alpha+\eta)^{\frac{3}{2}}}, \\ H_2 &= \frac{-4\alpha^2\left(8\alpha^2-4u_0\alpha+8\alpha\eta-u_0\eta+2\eta^2+(u_0-2\eta)\eta\cosh 2\alpha\rho\right)}{(4\alpha+\eta-\eta\cosh 2\alpha\rho)^2} \end{aligned} \tag{34}$$

etc.

4. Conclusion

We may hope to find the relationship between the conserved quantities and pss. The conserved quantities play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics [35]. In addition, the conserved quantities are a rich source of integrable systems suggested by the fact that they are the compatibility condition of an associated linear problem which admits enormous freedom if one allows the associated gauge algebra to be arbitrary [36].

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