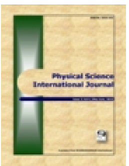

**Physical Science International Journal**
**4(3): 345-354, 2014**

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# The Weighed Average Geodetic of Distributions of Probabilities in the Statistical Physics

**V.V. Ryazanov**[*1]

[1] *Institute for Nuclear Research, Kiev, pr. Nauki, 47, Ukraine.*

***Original Research Article***

*Received: 20 March 2013*
*Accepted: 02 August 2013*
*Published: 20 November 2013*

## Abstract

The results obtained in [(1; 2)] for the statistical distributions at studying algebra of decision rules and natural geometry generated by it, are applied to estimations of the nonequilibrium statistical operator and superstatistics. The expressions for the nonequilibrium statistical operator and superstatistics are derived as special cases of the weighted geodetic average of the probability distributions.

*Keywords: weighted geodetic average of probability distributions, nonequilibrium statistical operator, superstatistics*

PACS: 05.70.Ln, 05.40.-a

## 1 Introduction

In [(1; 2)] the differential geometry of varieties of probabilistic measures which gives a natural language as the description of statistical model - to the a priori information on statistical experiment, and constructions of optimum methods of processing of such experiment is investigated. It is possible to interpret many results of works [(1; 2)] in terms of statistical physics. It concerns to exponent families of distributions, to "spread" of singular measures on all convex bearer, to problems of projecting, inequalities of the information, and other features of behaviour of the probabilistic distributions studied in [(1; 2)]. In the present work communication of the weighed average geodetic of continuous family of probabilistic laws [(1; 2)] with the nonequilibrium statistical operator (NSO) [(3; 4; 5)] and with superstatistics [(6; 7)] is traced.

**Corresponding author: E-mail: vryazan@yandex.ru*

# 2 The weighted geodetic average of the probability distributions

Following [(1)], we shall describe a class of probabilistic families for which in [(1; 2)] the notion of the weighted geodetic average is defined. We consider, that the smooth family $\overrightarrow{\Phi}$ of probabilistic laws can be described by means of the unique open map $(\Theta, \varphi)$, $\overrightarrow{x} = \varphi(P)$, or inverse mapping $\Theta \xrightarrow{\Psi} Caph(\Omega, \overrightarrow{S}, \overrightarrow{Z})$, where $\Omega$ is the space of all elementary outcomes $\omega$ (of an experiment), $\overrightarrow{S}$ is some $\sigma$-algebra of its subsets named also "the events". For each measure $\mu\{\cdot\}$ on the measurable space $(\Omega, \overrightarrow{S})$ all sets of zero measures (zero-sets) form an ideal $\overrightarrow{Z} = \overrightarrow{Z_\mu}$ of the algebra $\overrightarrow{S}$. The set of all probabilistic measures vanishing on the ideal $\overrightarrow{Z}$ and only on $\overrightarrow{Z}$, is designated through $Caph(\Omega, \overrightarrow{S}, \overrightarrow{Z})$. They form a subset of mutually absolutely continuous distributions on $(\Omega, \overrightarrow{S})$. If two measures $\mu$ and $\nu$ have the general ideal zero-sets they are called mutually absolutely continuous (or quasi-equivalent). If $\overrightarrow{Z_\mu} \subseteq \overrightarrow{Z_\nu}$ then it is said, that $\mu$ dominates $\nu$, and it is written as $\mu \gg \nu$. We call [(1)] the $m$-dimensional open map of set $M$ the one-to-one mapping $\varphi$ of the subset $\Theta \subseteq M$ on the coherent open area of an $m$-dimensional Euclidian space $R^m$. The coordinates $x^{(1)}(P)\ldots, x^{(m)}(P)$ of a point $\varphi(P)$ are thus referred to as local coordinates of a point $P \in M$ on the map $(\Theta, \varphi)$ under consideration. The surface $\{P_{\overrightarrow{x}}, \overrightarrow{x} \in \Theta\}$ in the manifold $Caph(\Omega, \overrightarrow{S}, \overrightarrow{Z})$ has no self-crossings, i.e. the mapping $\Psi = \varphi^{-1}$ is biunique. The described families were called simple families in [(1)].

The simple family of distributions of probabilities $\{P_{\overrightarrow{x}}, \overrightarrow{x} \in \Theta\} \subset Caph(\Omega, \overrightarrow{S}, \overrightarrow{Z})$ is called smooth [(1)], when

$1^\circ$ there is a coordinated variant of densities $p(\omega; \overrightarrow{x})$ on the fixed measure $\mu \in Conh(\Omega, \overrightarrow{S}, \overrightarrow{Z})$ such, that at every $\omega \in \Omega$ the density $p(\omega; \overrightarrow{x})$ is a three times differentiable positive function of the argument $(x_1 ..., x_n) = \overrightarrow{x} \in \Theta$; $(Conh(\Omega, \overrightarrow{S}, \overrightarrow{Z})$ is a set of all non-negative mutually absolutely continuous measures on $(\Omega, \overrightarrow{S})$, turning to zero on $\overrightarrow{Z}$-sets and only on it);

$2^\circ$ at every $\overrightarrow{x} \in \Theta$ the partial derivatives $p'_j(\omega; \overrightarrow{x}) = \partial p(\omega; \overrightarrow{x})/\partial x_j$ of density, $j = 1\ldots, n$, are linearly independent on $\Omega$ even neglecting their values on any $\overrightarrow{Z}$-set;

$3^\circ$ for every $\theta \in \Theta$ there is a special vicinity $O_\theta$, in which the derivatives $p\prime_j(\omega; \overrightarrow{x})$ allow a majorant $g(\theta)(\omega) = (dG(\theta)/d\mu)(\omega)$:

$$\left.\begin{array}{l} p'_j(\omega; \overrightarrow{x}) \le g^{(\theta)}(\omega), \quad P'_j\{\cdot \mid \overrightarrow{x}\} \le G^{(\theta)}\{\cdot\}, \quad \forall \overrightarrow{x} \in O_\theta, \quad \forall \omega \in \Omega, \\ M_{\overrightarrow{x}}[g^\theta(\omega)/p(\omega; \overrightarrow{x})]^2 \le L_\theta^2 < \infty, \quad \forall \overrightarrow{x} \in O_\theta \end{array}\right\}; \tag{2.1}$$

$4^\circ$ for every $\theta \in \Theta$ in the specified special vicinity $O_\theta$ all partial derivatives of likelihood function $\ln p(\omega; \overrightarrow{x})$ up to the third order inclusive allow a majorant

$$\left|\frac{\partial^{|\overrightarrow{k}|} \ln p(\omega; \overrightarrow{x})}{\partial \overrightarrow{x}^{\overrightarrow{k}}}\right| \le h^{(\theta)}(\omega), \quad \forall \overrightarrow{x} \in O_\theta, \quad |\overrightarrow{k}| = 1, 2, 3, \tag{2.2}$$

where $\overrightarrow{k} = (k_1, ..., k_n), \quad |\overrightarrow{k}| = k_1 + ... + k_n, \quad \partial \overrightarrow{x}^{\overrightarrow{k}} = \partial x_1^{k_1} ... \partial x_n^{k_n}$, and

$$M_{\overrightarrow{x}}[h^\theta(\omega)]^4 \le H_\theta^4 < \infty, \quad \forall \overrightarrow{x} \in O_\theta. \tag{2.3}$$

The constant $L_\theta^2$ from (2.1) is defined through estimations of the second derivatives.

In what follows we use the notations

$$\frac{\partial \ln p(\omega; \overrightarrow{x})}{\partial x_j} = r^j(\omega; \overrightarrow{x}), \qquad \frac{\partial^2 \ln p(\omega; \overrightarrow{x})}{\partial x_j \partial x_k} = r^{jk}(\omega; \overrightarrow{x}), \tag{2.4}$$

and the following consequence from the lemma 27.5 of [(1)]:

For any smooth family the identity holds:

$$M_{\overrightarrow{x}} r^j(\omega; \overrightarrow{x}) = 0; \tag{2.5}$$

$$-M_{\overrightarrow{x}}r^{jk}(\omega;\overrightarrow{x}) = M_{\overrightarrow{x}}r^{j}(\omega;\overrightarrow{x})r^{k}(\omega;\overrightarrow{x}) = \overrightarrow{\omega}^{jk}(\overrightarrow{x}), \tag{2.6}$$

where $M_{\overrightarrow{x}}r^{jk}(\omega;\overrightarrow{x}) = \int_\Omega r^{jk}(\omega;\overrightarrow{x})p(\omega;\overrightarrow{x})\mu\{d\omega\}$ is averaging, $\overrightarrow{\omega}_{jk}(\theta) = M_\theta r^j(\omega;\theta)r^k(\omega;\theta)$ is the Fisher information matrix. Alongside with initial parametrization of family we shall consider also its linear reparametrization. When in a new system of coordinates the Fisher information matrix in a point $\theta$ is the unit matrix such system of coordinates in [(1)] is called $\theta$-local. In [(1)] the $\theta$-local distance between the distribution laws $P_{\overrightarrow{x}}$ and $P_{\overrightarrow{\tau}}$ is considered:

$$||\overrightarrow{x}-\overrightarrow{\tau}||^2_\theta = \sum_{j,k}(x_j-\tau_j)(x_k-\tau_k)\overrightarrow{\omega}^{jk}(\theta). \tag{2.7}$$

Introduce now according to [(1)] the notion of the weighted geodetic average of the continuous family of probability laws. Let the family $\overrightarrow{\Phi} = \{P_{\overrightarrow{x}}, \overrightarrow{x} \in C\}$ of distributions on $(\Omega, \overrightarrow{S})$, depending on the vector parameter $\overrightarrow{x}$ with compact set $C$ values of a parameter, is defined by a family of coordinated strictly positive densities $p(\omega;\overrightarrow{x})$ with respect to the measure $R$, continuous on $\overrightarrow{x}$ at every $\omega \in \Omega$. Let $\alpha\{\cdot\}$ be any probabilistic Borel measure on $C$. In (1) the weighed (with a weight measure $\alpha$) geodetic average of laws of family $\overrightarrow{\Phi}$ is the probability distribution $U_\alpha$ with the logarithm of density

$$\ln u_\alpha(\omega) = \int_C \ln p(\omega;\overrightarrow{x})\alpha\{d\overrightarrow{x}\} - H[\alpha], \tag{2.8}$$

where $H[\alpha]$ is the logarithm of a normalizing divider

$$expH[\alpha] = \int_\Omega \exp[\int_C \ln p(\omega;\overrightarrow{x})\alpha\{d\overrightarrow{x}\}]R\{d\omega\}, \tag{2.9}$$

if only last integral is finite. Otherwise we consider, that the specified average does not exist.

In [(1)] the set $\gamma \subset Caph(\Omega, \overrightarrow{S}, \overrightarrow{Z})$ of probability distributions $P_s\{\cdot\}$ of exponent or geodetic family (finite number of measurements) with canonical affine parameter $\overrightarrow{s} = (s_1\dots,s_n)$ is introduced with the family of densities

$$\frac{dP_s}{d\mu}(\omega) = p(\omega;\overrightarrow{s}) = p_0(\omega)\exp[\sum_j s^j q_j(\omega) - \Psi(s)]\,, \tag{2.10}$$

where $\overrightarrow{q} = (q_1(\omega)\dots,q_n(\omega))$ is a directing sufficient statistics (1), $\mu\{\cdot\}$ is the fixed dominating measure, and

$$\exp[\Psi(s)] = \int_\Omega \exp[\sum_j s^j q_j(\omega)]p_0(\omega)\mu\{d\omega\} \tag{2.11}$$

is a normalizing divider. It is supposed, that the parameter $\overrightarrow{s}$ of the distribution takes values at which the normalizing divider is finite, i.e. $\gamma$ is the maximal family of distributions, representable at the some $s^1\dots,s^n$ in the form of (2.10). In [(1; 2)] it is shown, that the family of densities in (2.10) - (2.11) is a "trajectory" of a $n$-dimensional subgroup of group of translations of the manifold $Caph(\Omega, \overrightarrow{S}, \overrightarrow{Z})$ of the probability distributions. The distributions (2.10) are included into a wider class of exponential families with the density in the form

$$p(\omega;\overrightarrow{\theta}) = p_0(\omega)exp[\sum_j s^j(\overrightarrow{\theta})q_j(\omega) - \Psi(\overrightarrow{s}(\overrightarrow{\theta}))], \tag{2.12}$$

where $\overrightarrow{\theta} = (\theta_1,\dots,\theta_m) \in \Theta, \overrightarrow{s}(\overrightarrow{\theta}) = (s^1(\overrightarrow{\theta}),\dots,s^n(\overrightarrow{\theta}))$.

Two values of parameters in [(1; 2)] are related with every geodetic average $U_\alpha$ with density (2.8) weighed with weight $\alpha\{\cdot\}$:

$$\overrightarrow{X}[\alpha] = \int_C \overrightarrow{x}\alpha\{d\overrightarrow{x}\}, \tag{2.13}$$

$$\overrightarrow{Y}[\alpha] : Y_j = y_j + M_{\overrightarrow{y}}[\ln u_\alpha(\omega) - \ln p(\omega; \overrightarrow{y})] \sum_k v_{jk}(\overrightarrow{y}) r^k(\omega; \overrightarrow{y}), \tag{2.14}$$

where $r^k(\omega; \overrightarrow{z})$ is defined in (2.4) - (2.5), a matrix $\overrightarrow{V}(\overrightarrow{x}) = (\overrightarrow{v}_{jk})(\overrightarrow{x})$ is the inverse of the information matrix $\overrightarrow{W}(\overrightarrow{x}) = (\overrightarrow{w^{jk}})(\overrightarrow{x})$ (2.6). When $\overrightarrow{Y}[\alpha] \in \Theta$, it is called that the law $P_{\overrightarrow{Y}[\alpha]}$ is an accompaniment of the weighed geodetic average $U_\alpha$. The point $\overrightarrow{y} \in F \subset \Theta$ is set in [(1)] as the center of a cube

$$C_r = \{\overrightarrow{x} : |x_j - y_j| \le r; j = 1, ..., n = dim\overrightarrow{\Phi}\} \tag{2.15}$$

in the space of $\overrightarrow{y}$-local parameters of a smooth compact family $\overrightarrow{\Phi} = \{P_{\overrightarrow{x}}, \overrightarrow{x} \in F\}; rn^{1/2} \le \rho(\Phi)$; all cube (2.15) belongs to the compact set $K(\overrightarrow{\Phi}) \subset \Theta$, and the uniform estimates of derivatives (2.1) - (2.3) are satisfied. The corresponding family $\{P_{\overrightarrow{x}}, \overrightarrow{x} \in C_r\}$ is denoted $\overrightarrow{\Phi}(r) = \overrightarrow{\Phi}_{\overrightarrow{y}}(r)$ in [(1)] and called cubic.

In [(1; 2)] it is proven, that

$$||\overrightarrow{Y}[\alpha] - \overrightarrow{X}[\alpha]||_y \le r^2 H^2 n^{3/2}, \tag{2.16}$$

where $H$ is a constant from (2.3). For the family $\overrightarrow{\Phi}$ it is possible to specify the size $\rho_0(\overrightarrow{\Phi})$ such, that at $r < \rho_0(\overrightarrow{\Phi})$ there exists the accompanying law $P_{Y[\alpha]}$, for every $\overrightarrow{y} \in F$ and probabilistic measure $\alpha\{\cdot\}$, and there holds $Y[\alpha] \in C_{2r} \subset K(\overrightarrow{\Phi})$.

For the information deviations (Kullback entropy)

$$I[Q|P] = \int_\Omega [\frac{dP}{dQ}(\omega) \ln \frac{dP}{dQ}(\omega)] Q\{d\omega\} = -\int_\Omega [\ln \frac{dQ}{dP}(\omega)] P\{d\omega\} = \int_\Omega [\ln \frac{dP}{dQ}(\omega)] P\{d\omega\} \tag{2.17}$$

(last equality holds, when the probability laws $P$ and $Q$ in (2.17) are mutually absolutely continuous) the relation is then written:

$$I[P|U] = I[P|P_Y] + I[P_Y|U] + \langle \ln(dP/dP_Y), U - P_Y \rangle, \tag{2.18}$$

where $P_Y$ is the accompanying law, $\langle f, P \rangle = \int f(\omega) P\{d\omega\}$, $U \in \Gamma(N) = \Gamma(\overrightarrow{\Phi}_N)$, $\Gamma(\overrightarrow{\Phi}_N)$ is an integrated convex envelope of the initial family $\overrightarrow{\Phi}$ [(1)], that is the family of probabilistic laws with densities (2.8), containing a convex envelope of the initial family $\overrightarrow{\Phi}$, $\overrightarrow{\Phi}_N = \overrightarrow{\Phi}(r(N)), r(N) = N^{-3/2} < \rho_0(\overrightarrow{\Phi})$.

In [(1; 2)] the difference between $\ln u_\alpha(\omega)$ and $\ln p(\omega; \overrightarrow{x})$ and $u_\alpha(\omega)$ and $p(\omega; \overrightarrow{x})$ was also estimated. For the cubic family $\overrightarrow{\Phi}(r)$ at $r < \rho_0(\overrightarrow{\Phi})$ the weighted distribution $U_\alpha$ is close to the accompanying law $P_{Y[\alpha]}$:

$$|\ln u_\alpha(\omega) - \ln p(\omega; \overrightarrow{Y})| \le r^2 [B_2 + B_3 h^{(y)}(\omega)], \tag{2.19}$$

$$|u_\alpha(\omega) - p(\omega; \overrightarrow{Y})| \le r^2 g^{(y)} [B_2 + B_3 h^{(y)}(\omega)] B_4, \tag{2.20}$$

where $B_2 = B_2(n, H) = 4nH, B_3 = B_3(n, H) = 4n + Hn^{3/2}, B_4 = \exp[4n\rho_0 H]$, values $h^{(y)}, g^{(y)}, r, H$ are defined in (2.1) - (2.3), (2.15). In [(1)] the conditions of convergence of distributions (2.8) and for compactness of the integrated convex envelope of the family $\overrightarrow{\Phi}$ are derived as well.

# 3 Nonequilibrium Statistical Operator as the weighed geodetic average for the probabilistic laws of the quasi-equilibrium distributions family

In [(8)] the logarithm of NSO $\rho(t)$ [(3; 4; 5)] is interpreted as an averaging of the logarithm of the quasi-equilibrium distribution $\rho_q$ [(4; 5)] as function of different time arguments on the distribution $p_q(u)$ of the system lifetime (defined as the time of the first achievement of a certain level):

$$\ln \rho(t) = \int_0^\infty p_q(u) \ln \rho_q(t-u, -u) du, \tag{3.1}$$

where $u = t - t_0$ is a random variable of a the lifetime of a system, $t$ is the present time moment, $t_0$ is a random variable of the initial time moment, i.e. the timemark of the system "birth". The value $u = t - t_0$ is equal to the random moment of the first achievement of a zero level [(9; 10)] during the moment $t_0$ in return time, at $t \mapsto -t$, (3.2).

$$\Gamma_x = inf\{t : y(t) = 0\}, \quad y(0) = x > 0. \tag{3.2}$$

If $p_q(u) = \varepsilon e^{\varepsilon u}$, the distribution $p_q(u)$ has an exponent form with $\varepsilon = 1/\langle\Gamma\rangle$, where $\langle\Gamma\rangle = \langle t - t_0\rangle$ is the average of the system lifetime, from (3.1) the NSO in the form proposed by Zubarev [(3; 4; 5)] is recovered. The quasi-equilibrium distribution $\rho_q$ reads as [(4)]:

$$\ln \rho_q(t_1, t_2) = -\Phi(t_1) - \sum_n F_n(t_1) P_n(t_2), \tag{3.3}$$

where the dependence $P_n(t_2)$ is understood as a realization of the conservation laws [(3)] when the operators $P_n$ in the quantum case are considered in Heisenberg representation, and in the case of classical mechanics the Heisenberg representation is replaced with the action of the evolution operator, for example

$$H(x,t) = e^{-iLt} H(x); \quad \rho_q(t-u, -u) = e^{-iuL} \rho_q(t-u, 0), \tag{3.4}$$

where $L$ is the Liouville operator (3). The values $P_n$ in (3.3) represent dynamic variables (for example, energy, number of particles, etc.); their average values give a set of observable values; $F_n$ are Lagrange multipliers related to the intensive thermodynamic variables (temperature, chemical potential, etc). Similar expressions encounter not only for the hydrodynamical, but also for the kinetic stage of the system evolution [(4; 5)].

The expressions (3.3) for $\rho_q$ correspond to the exponential family (2.12) and coincide with it at

$$\rho_q(t; \omega) = p(\omega; \overrightarrow{\theta})/p_0(\omega); \quad \overrightarrow{\theta} = \overrightarrow{x} = u = t - t_0, \tag{3.5}$$
$$\Phi(t-u) = \Psi(\overrightarrow{s}(\overrightarrow{\theta})), \quad P_n = q_n(\omega); \quad F_j(t-u) = -s^j(\overrightarrow{\theta}).$$

If the conditions (3.5) are satisfied the expression for NSO (3.1) coincides with (2.8) at

$$\alpha\{d\overrightarrow{x}\} = p_q(u) du, \quad u = \overrightarrow{x} = t - t_0 = \Gamma, \quad H[\alpha] = 0. \tag{3.6}$$

Let's show, that for NSO (3.1) $H[\alpha] = 0$. The value $H[\alpha]$ in (2.8) - (2.9) for the distribution $p(\omega, \overrightarrow{x})$ of the form (2.12) if the conditions (3.5) - (3.6) are satisfied, reads as

$$H[\alpha] = \Psi(\int_C \alpha\{d\overrightarrow{x}\} s^j(\overrightarrow{x}) q_j(\omega)) - \int_C \alpha\{d\overrightarrow{x}\} \Psi(s^j(\overrightarrow{x}) q_j(\omega)), \tag{3.7}$$

where $\Psi(s)$ is defined in (2.11). In [(3)] where Zubarev's NSO $\rho_Z$ corresponds to an invariant part from the logarithm of the locally-equilibrium operator $\rho_l$ (or $\rho_q$) (3), i.e.

$$\ln \rho_Z(t) = \varepsilon \int_0^\infty e^{-\varepsilon u} \ln \rho_l(t-u,-u)du, \tag{3.8}$$

for

$$\Phi_l(t-u) = \Psi(s(t-u)) = \ln Sp \exp\{-\sum_m \int_V F_m(\overrightarrow{r}, t-u) P_m(\overrightarrow{r}, -u) d\overrightarrow{r}\},$$

where $\overrightarrow{r}$ is spatial coordinate, $V$ is the volume of system, the dependence $P_m$ from $u$ is given in (3.4), the relations hold:

$$\Phi_l = \varepsilon \int_0^\infty e^{-\varepsilon u} \Phi_l(t-u)du = \varepsilon \int_0^\infty due^{-\varepsilon u} \ln Sp \exp\{-\sum_m \int_V F_m(\overrightarrow{r}, t-u) P_m(\overrightarrow{r}, -u) d\overrightarrow{r}\};$$

$$\Phi_l = \ln Sp \exp\{-\sum_m \varepsilon \int_0^\infty \int_V e^{-\varepsilon u} F_m(\overrightarrow{r}, t-u) P_m(\overrightarrow{r}, -u) dud\overrightarrow{r}\}.$$

Substituting these expressions in (3.7), we obtain $H[\alpha] = 0$.

The expression (2.13) for the case of NSO with relations (3.5) - (3.6) satisfied, defines the average lifetime of system, and the expression (2.14) gives an estimation for it, (named the quasi-projection in terms of [(1; 2)]) with accuracy of (2.16). The expressions (2.4) - (2.5) coincide with the operator of entropy production [(3; 4; 5)] $\hat{\sigma}(t-u,-u) = \partial \ln \rho_q(t-u,-u)/\partial u$.

In [(8)] estimations of a kind (2.14) for a cube (2.15) with the center in a point $\overrightarrow{y} = 0$

$$Y[\alpha] = \int_\Omega [\ln \rho(t) - \ln \rho_q(t,o)] \frac{\hat{\sigma}(t,0)\rho_q(t,o)}{\langle \hat{\sigma}^2(t,0)\rangle_q} dz, \tag{3.9}$$

where $z = (q_1 \dots, q_N; p_1 \dots, p_N)$ is set of coordinates $q$ and impulses $p$ all particles of system, $z = \omega$ in (2.1) - (2.6); $\langle \dots \rangle_q = \int_\Omega \dots \rho_q(t,o)dz$, are compared to the similar expressions derived directly from NSO. In [(8)] the example of calculation of lifetime average for a system of neutrons in a nuclear reactor is performed.

Part of the expression (3.9) with $Y \sim \langle \Gamma \rangle$, are the entropy production $\hat{\sigma}$ and entropy fluxes [(3; 4; 5)]. At $\hat{\sigma} \to 0, \langle \Gamma \rangle \sim \frac{0}{0^2} \to \infty$, and at $\hat{\sigma} \to \infty, \langle \Gamma \rangle \sim \frac{\infty}{\infty^2} \to 0$. Thus, a lifetime of the system depends on the entropy production and entropy fluxes in a system due to exchange of entropy between the system and the environment.

Integrating in parts the expression (3.1), we obtain, that at $\int p_q(u)du_{|u=0} = -1, \int p_q(u)du_{|u\to\infty} = 0$, [(8)],

$$\ln \rho(t) = \ln \rho_q(t,0) - \int_0^\infty (\int p_q(u)du) \hat{\sigma}(t-u,-u)du. \tag{3.10}$$

From here

$$-\int_0^\infty (\int p_q(u)du)\hat{\sigma}(t-u,-u)du = \ln \rho(t) - \ln \rho_q(t-Y,-Y) + \ln \rho_q(t-Y,-Y) - \ln \rho_q(t,0).$$

The first term in the right part of the obtained expression, value $\ln \rho(t) - \ln \rho_q(t-Y,-Y)$ can be estimated by means of the relation (2.19) provided that the conditions (3.5) - (3.6) are satisfied, when $u_\alpha(\omega) = \rho(t), p(\omega,x) = \rho_q(t-x,-x)$. The second term is estimated by means of the relation obtained in [(1)]:

$$\ln[p(\omega;x)/p(\omega;Y)] \lesseqgtr \pm ||x-Y|| h^{(y)}(\omega)\,,$$

where $h^{(y)}$ is taken from (2.2). Then

$$-\int_0^\infty (\int p_q(u)du)\hat{\sigma}(t-u,-u)du \leq r^2[4nH + (4n + Hn^{3/2})h^{(y)}(\omega)] + \rho(\Phi)h^{(y)}(\omega).$$

In the left part the value of $\hat{\sigma}$ is estimated from the relation (2.2), and

$$-\int_0^\infty (\int p_q(u)du)\hat{\sigma}(t-u,-u)du \leq -h^{(y)}(\omega)\int_0^\infty (\int p_q(u)du)du = h^{(y)}(\omega)\langle\Gamma\rangle.$$

Thus, the lifetime average $\langle\Gamma\rangle$ is limited, and in (3.8) $\varepsilon = 1/\langle\Gamma\rangle \neq 0$ though in [(3; 4; 5)] a limiting transition $\varepsilon \to 0$ after the thermodynamic limiting transition is performed. The reason for it is that in [(1; 2)] a cubic family in the limited cube (2.15) is considered with lifetimes limited by the value $r$. In the theory of NSO [(3; 4; 5)] the limiting transition $\varepsilon \to 0$ is carried out after thermodynamic limiting transition $V \to \infty, N \to \infty, V/N = const$. Intuitively it is clear, that the lifetime of an infinite large systems will be infinitely large as well.

Similar estimations hold for the expression (2.20) in view of the relation derived from (3.10)

$$\rho(t) - \rho_q(t,0) = \sum_{k=1}^{\infty}[-\int_0^\infty du(\int p_q(u)du)\hat{\sigma}(t-u,-u)du]^k \rho_q(t,0).$$

# 4 Superstatistics as the weighed geodetic average of probabilistic laws of Gibbs family distributions depending on a nonequilibrium parameter

The distributions of a kind (2.8) describe not only NSO. In works [(6; 7)] the superstatistics of A type are introduced with:

$$p(E) = B(E)/Z_A; \quad B(E) = \int_0^\infty f(\beta)\exp\{-\beta E\}d\beta; \quad Z_A = \int_0^\infty B(E)\omega(E)dE, \qquad (4.1)$$

where $f(\beta)$ is some distribution of value $\beta$, inverse temperature, the intensive thermodynamic variable, the conjugate of energy $E$, and B type, with

$$p(E) = \int_0^\infty f(\beta)\frac{\exp\{-\beta E\}}{Z(\beta)}d\beta; \quad Z(\beta) = \int_0^\infty \exp\{-\beta E\}\omega(E)dE; \quad \int_0^\infty f(\beta)d\beta = 1. \qquad (4.2)$$

The expression (4.1) passes into (4.2) after replacement $\tilde{f}(\beta) = \frac{cf(\beta)}{Z(\beta)}$, $c = const$. The special case of superstatistics, at function $f(\beta)$, set in the form of gamma-distribution, leads to Tsallis distributions [(11)] with $\beta_0 = \int \beta f(\beta)d\beta$.

If in (4.1) instead of the distribution $p(\beta; E) = exp\{-\beta E\}/Z(\beta)$ the distribution (2.12) is used with $\overrightarrow{x} = \overrightarrow{\theta}$ and $p(\omega; \overrightarrow{x}) = p(\omega; \overrightarrow{\theta})/p_0(\omega) = p(\beta(\theta); E)$ for a case A from (4.1), substituting (2.12) in (2.8), we obtain the coincidence with (4.1) at

$$s(\theta) = -\beta(\theta), \quad q = E; \quad \ln(p(\omega; \overrightarrow{\theta})/p_0(\omega)) = -\beta(\theta)E - \ln Z(\beta(\theta)); \qquad (4.3)$$

$$u_\alpha(\omega) = p(E); \quad Z(\beta(\theta)) = \int \exp[-\beta(\theta)E]\omega(E)dE;$$

$$\int \ln p(\beta(\theta); E)\alpha(\theta)d\theta = -\int \beta(\theta)\alpha(\theta)d\theta E - \int \ln Z(\beta(\theta))\alpha(\theta)d\theta =$$

$$\ln \int_0^\infty f(\beta)\exp\{-\beta E\}d\beta = \ln B(E).$$

In this case $H[\alpha] = \ln Z_A \neq 0$, unlike (3.6). For the value of $H[\alpha]$ the estimations are given in [(1; 2)]:

$$0 \leq -H[\alpha] \leq 4nr^2 H$$

in (1), and

$$-H[\alpha] \leq \int_C I[P_x|R]\alpha\{dx\}$$

in [(2)] where the existence of such distribution of probabilities $R(\cdot)$ on $(\Omega, \overrightarrow{S})$, is supposed, that $sup_C I[P_x|R] < \infty$ (designations $C, (\Omega, \overrightarrow{S})$ correspond to (2.8)). From (4.3) the relation between distribution functions $f(\beta)$ and $\alpha(\theta)$ is determined.

The parameter $\theta$ in (4.3) represents some extensive thermodynamic parameter corresponding to an internal thermodynamic parameter, describing nonequilibrium state of the system [(12)]. It can be the coordinate of the center of weights in the gravity field, the electric moment of dielectric in an external electric field (12), number of phase jumps in the problems of phase synchronization [(13; 14)], etc. The average values are

$$\langle\theta\rangle = \int \theta\alpha(\theta)d\theta; \quad \beta_\theta = \int \beta(\theta)\alpha(\theta)d\theta.$$

Generally $\beta_\theta$ does not coincide with $\beta_0$. But for big systems $\beta_\theta \sim \beta_0$. As $\overrightarrow{\theta}$ is the vector value expressions in the form (4.3), obtained from (2.12), (2.8), describes also superstatistics with several fluctuating thermodynamic parameters. Such expressions are derived in [(15; 16)].

The estimations of the Section 3 for NSO, can be applied for the superstatistics as well. So, for the A type superstatistics the relation (2.19) is written in the form of ( $n = 1$):

$$\ln(\int_0^\infty f(\beta)e^{-\beta E}d\beta) + \beta(\theta[\alpha])E + \ln Z(\beta(\theta[\alpha])) - \ln\int_0^\infty(\int_0^\infty f(\beta)e^{-\beta E}d\beta)\omega(E)dE \leq$$

$$r^2[4H + (4+H)h^{(y)}(\omega)],$$

where ( (2.2), (2.3), (2.14))

$$\frac{\partial \ln p(\omega;\theta)}{\partial\theta} = \frac{\partial \ln p(\omega;\theta)}{\partial\beta(\theta)}\frac{\partial\beta(\theta)}{\partial\theta} = \frac{\partial\beta(\theta)}{\partial\theta}[\langle E(\beta(\theta))\rangle - E] \leq h^{(y)}(\omega); \quad \langle E(\beta(\theta))\rangle = -\frac{\partial \ln Z(\beta(\theta))}{\partial\beta(\theta)};$$

$$\int(\frac{\partial\beta(\theta)}{\partial\theta})^4[\langle E(\beta(\theta))\rangle - E]^4\frac{e^{-\beta(\theta)E}}{Z(\beta(\theta))}\omega(E)dE \leq M_\theta[h^{(y)}(\omega)]^4 \leq H^4;$$

$$\theta[\alpha] = y + \int \ln B(E)\frac{[\langle E(\beta(y))\rangle - E]}{\frac{\partial\beta(y)}{\partial y}\frac{\partial^2 \ln Z(\beta(y))}{\partial\beta^2(y)}}\frac{e^{-\beta(y)E}}{Z(\beta(y))}\omega(E)dE - \beta(y)\frac{1}{\frac{\partial\beta(y)}{\partial y}}.$$

If the center of the cube $C_r$ (2.15) is located in a point $y = 0$, $\theta \leq r$. To estimate an arrangement of the center of the cube and the value of the parameter $r$, it is necessary to know the physical nature of the parameter $\theta$, relating the consideration to the specific physical situation.

The expression (2.20) for the A type superstatistics and the relations (4.1) turn to

$$|\frac{B(E)}{Z_A} - \frac{e^{-\beta(\theta[\alpha])E}}{Z(\beta(\theta[\alpha]))}| \leq r^2 g^{(y)}(\omega)[4nH + h^{(y)}(\omega)(4n + Hn^{3/2})]e^{4n\rho_0 H}.$$

The value $\rho_0$, defined before the expression (2.17), can be also estimated after setting the parameter $\theta$.

# 5 Conclusion

In this paper we performed a general estimate of the differences of the nonequilibrium statistical operator of the quasi-equilibrium statistical operator for arbitrary distribution functions of the system lifetime. The estimations of the average system lifetime are given. This value is dependent on the entropy production and entropy fluxes, due to entropy exchange between the system and its environment. The differences between the type A superstatistics distribution and Gibbs distributions are discussed.

The papers [(1; 2)] contain a number of results important for the statistical physics. The results of sections 3-5 of the present work are formulated by means of the projective methods developed in (1; 2) which importance is emphasized in the theory of NSO [(3; 4; 5)]. The problem A of projecting [(1; 2)] corresponds to finding the minimum of the Kullback entropy (2.17) for the nonequilibrium system [(17)], §29.5. The expressions for the divergence of Amari, Kagan, Csiszar [(2)] are compared to the entropy functionals, used in Tsallis statistics [(11)] (for example, to the Renyi information quantities). In [(1; 2)] the methods are developed allowing a rigorous approach to the important problem in NSO of selection of basic variables of the quasi-equilibrium distribution [(3; 4; 5)]. An interesting problem is presented for finding an interpretation in the statistical physics of such concepts as statistical decision rules, risk assessment or asymmetrical pythagorean geometry [(1)].

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## Competing Interests

The author declares that no competing interests exist.

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