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Qualitative analysis of solutions for a parabolic type Kirchhoff equation with logarithmic nonlinearity

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Abstract: In this work, we investigate the initial boundary-value problem for a parabolic type Kirchhoff equation with logarithmic nonlinearity. We get the existence of global weak solution, by the potential wells method and energy method. Also, we get results of the decay and finite time blow up of the weak solutions.

Keywords: Blow up; Global existence; Logarithmic nonlinearity; Parabolic type Kirchhoff equation.

MSC: 35A01; 35B40; 35K20.

1. Introduction

In this work, we investigate the existence of global, decay and finite time blow up of solutions for the parabolic type Kirchhoff equation with logarithmic source term

$$\begin{cases} u_t - M(\|\nabla u\|^2)\Delta u - \Delta u_t = u^{k-1}\ln|u| - \oint_\Omega u^{k-1}\ln|u|\,dx, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bound domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$. Also, $M(s) = 1 + s^\gamma$, $(\gamma > 0)$, $\oint_\Omega u_0 dx = \frac{1}{|\Omega|}\int_\Omega u_0 dx = 0$ and

$$\begin{cases} 2\gamma + 2 \leq k \leq +\infty, & n = 1,2, \\ 2\gamma + 2 \leq k \leq \frac{2n}{n-2}, & n \geq 3. \end{cases}$$

Many other authors studied the problem (1) in a more general form

$$\begin{cases} u_t - \Delta u = f(u) - \oint_\Omega f(u)dx, & x \in \Omega,\ t > 0, \\ \frac{\partial u(x,t)}{\partial \eta} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bound domain in $\mathbb{R}^n$ $(n \geq 1)$ with $|\Omega|$ denoting its Lebesgue measure, the function $f(u)$ is usually taken to be a power of u, and n is the outer normal vector of $\partial\Omega$. Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in different areas of physics such as supersymmetric field theories, optics, quantum mechanics and inflationary cosmology [1,2]. When $M \equiv 1$ and $k = 2$, (1) become semilinear pseudo-parabolic equation as follow

$$u_t - \Delta u - \Delta u_t = u \log|u|. \tag{2}$$

Chen and Tian [3] obtained the global existence, behavior of vacuum isolation and blow-up of solutions at $+\infty$ of the Equation (2). Without Δu, (2) become the following semilinear parabolic equation

$$u_t - \Delta u_t = u \log|u|. \tag{3}$$

Chen *et al.*, [4] studied the global existence, decay and blow-up at $+\infty$ of solutions of the Equation (3). Yan and Yang [5] studied nonlocal parabolic equation with logarithmic nonlinearity

$$u_t - \Delta u_t = u \log |u| - \oint_{\Omega} u \log |u| \, dx.$$

Recently, they obtained the results under appropriate conditions on blow-up and existion of the solutions. Toualbia *et al.*, [6] studied the initial boundary value problem of a nonlocal parabolic equation with logarithmic nonlinearity

$$u_t - div(|\nabla u|^{p-2} \nabla u) = |u|^{p-2} u \log |u| - \oint_{\Omega} |u|^{p-2} u \log |u| \, dx.$$

They obtained the decay, blow up and nonextinction of solutions under appropriate condition. Also, recently some authors studied the parabolic and hyperbolic type equation with logarithmic source term (see [7–15]).

The organization of the remaining part of this paper is as follows: In the next Section 2, we introduce some lemmas which will be needed later. In Section 3, under some conditions, we get the unique global weak solution of the problem (1). Moreover, we find that the decay of solutions. In the lastly, we get the blow up theorem.

2. Preliminaries

Throughout this work, we adopt the following abbreviations

$$\|u\|_s = \|u\|_{L^s(\Omega)}, \ \|u\|_{H_0^1(\Omega)} = \left(\|\nabla u\|^2 + \|u\|^2 \right)^{\frac{1}{2}},$$

for $1 < s < \infty$.

The energy functional J and Nehari functional I defined on $H_0^1(\Omega) \backslash \{0\}$ as follow

$$J(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} - \frac{1}{k} \int_{\Omega} |u|^k \ln |u| \, dx + \frac{1}{k^2} \int_{\Omega} |u|^k \, dx, \tag{4}$$

and

$$I(u) = \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} - \int_{\Omega} |u|^k \ln |u| \, dx. \tag{5}$$

By (4) and (5), we obtain

$$J(u) = \frac{1}{k} I(u) + \frac{k-2}{2k} \|\nabla u\|^2 + \frac{k-2\gamma-2}{2k(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} + \frac{1}{k^2} \int_{\Omega} |u|^k \, dx. \tag{6}$$

Let

$$\mathcal{N} = \{ u \in H_0^1(\Omega) \backslash \{0\} : I(u) = 0 \},$$

be the Nehari manifold. Thus, we may define

$$d = \inf_{u \in \mathcal{N}} J(u), \tag{7}$$

d is positive and is obtained by some $u \in \mathcal{N}$.

Lemma 1. *Let $u \in H_0^1(\Omega) \backslash \{0\}$, we consider the function $j : \lambda \to J(\lambda u)$ for $\lambda > 0$. Then we possess*

(i) $\lim_{\lambda \to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda \to +\infty} j(\lambda) = -\infty$;
(ii) there is a unique $\lambda^* > 0$ such that $j'(\lambda^*) = 0$;
(iii) $j(\lambda)$ is strictly increasing on $(0, \lambda^*)$, strictly decreasing on $(\lambda^*, +\infty)$ and takes the maximum at λ^*; $I(\lambda u) = \lambda j'(\lambda)$ and

$$I(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty. \end{cases}$$

Proof. For $u \in H_0^1(\Omega)\backslash\{0\}$, by the definition of j, we get

$$\begin{aligned} j(\lambda) &= \frac{1}{2}\left\|\nabla(\lambda u)\right\|^2 + \frac{1}{2(\gamma+1)}\left\|\nabla(\lambda u)\right\|^{2(\gamma+1)} - \frac{1}{k}\int_\Omega |\lambda u|^k \ln|\lambda u|\,dx + \frac{1}{k^2}\int_\Omega |\lambda u|^k\,dx \\ &= \frac{\lambda^2}{2}\left\|\nabla u\right\|^2 + \frac{\lambda^{2(\gamma+1)}}{2(\gamma+1)}\left\|\nabla u\right\|^{2(\gamma+1)} - \frac{\lambda^k}{k}\int_\Omega |u|^k \ln|u|\,dx - \frac{\lambda^k}{k}\int_\Omega |u|^k \ln\lambda dx + \frac{\lambda^k}{k^2}\int_\Omega |u|^k\,dx. \qquad (8) \end{aligned}$$

It is clear that *(i)* holds due to $\int_\Omega |u|^k\,dx \neq 0$. We get

$$\begin{aligned} \frac{d}{d\lambda}j(\lambda) =&\lambda\left\|\nabla u\right\|^2 + \lambda^{2\gamma+1}\left\|\nabla u\right\|^{2(\gamma+1)} - \lambda^{k-1}\int_\Omega |u|^k \ln|u|\,dx \\ &- \lambda^{k-1}\ln\lambda\int_\Omega |u|^k\,dx - \frac{\lambda^{k-1}}{k}\int_\Omega |u|^k\,dx + \frac{\lambda^{k-1}}{k}\int_\Omega |u|^k\,dx \\ =&\lambda\left\|\nabla u\right\|^2 + \lambda^{2\gamma+1}\left\|\nabla u\right\|^{2(\gamma+1)} - \lambda^{k-1}\int_\Omega |u|^k \ln|u|\,dx - \lambda^{k-1}\ln\lambda\int_\Omega |u|^k\,dx \\ =&\lambda\left(\left\|\nabla u\right\|^2 + \lambda^{2\gamma}\left\|\nabla u\right\|^{2(\gamma+1)} - \lambda^{k-2}\int_\Omega |u|^k \ln|u|\,dx - \lambda^{k-2}\ln\lambda\int_\Omega |u|^k\,dx\right). \end{aligned}$$

Let

$$\varphi(\lambda) := \lambda^{2\gamma}\left\|\nabla u\right\|^{2(\gamma+1)} - \lambda^{k-2}\int_\Omega |u|^k \ln|u|\,dx - \lambda^{k-2}\ln\lambda\int_\Omega |u|^k\,dx.$$

Then from $2\gamma+2 \leq k$ we can conclude that $\lim_{\lambda\to\infty}\varphi(\lambda) = -\infty$, $\varphi(\lambda)$ is monotone decreasing when $\lambda > \lambda^*$ and there exists a unique λ^* such that $\varphi(\lambda)\mid_{\lambda=\lambda^*=0}$. Hence, we obtain there is a $\lambda^* > 0$ such that $\left\|\nabla u\right\|^2 + \varphi(\lambda) = 0$, which means $\frac{d}{d\lambda}J(\lambda u)\mid_{\lambda=\lambda^*=0}$. The conclusion *(iii)* directly follows from the proof of the conclusion *(ii)* and

$$\begin{aligned} I(\lambda u) =& \left\|\nabla(\lambda u)\right\|^2 + \left\|\nabla(\lambda u)\right\|^{2(\gamma+1)} - \int_\Omega |\lambda u|^k \ln|\lambda u|\,dx \\ =&\lambda^2\left\|\nabla u\right\|^2 + \lambda^{2(\gamma+1)}\left\|\nabla u\right\|^{2(\gamma+1)} - \lambda^k\int_\Omega |u|^k \ln|u|\,dx - \lambda^k\int_\Omega |u|^k \ln\lambda dx \\ =&\lambda^2\left\|\nabla u\right\|^2 + \lambda^{2(\gamma+1)}\left\|\nabla u\right\|^{2(\gamma+1)} - \lambda^k\int_\Omega |u|^k \ln|u|\,dx - \lambda^k\ln\lambda\int_\Omega |u|^k\,dx \\ =&\lambda\left(\lambda\left\|\nabla u\right\|^2 + \lambda^{2\gamma+1}\left\|\nabla u\right\|^{2(\gamma+1)} - \lambda^{k-1}\int_\Omega |u|^k \ln|u|\,dx - \lambda^{k-1}\ln\lambda\int_\Omega |u|^k\,dx\right) \\ =&\lambda j'(\lambda). \end{aligned}$$

Thus, $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$. The proof is complete. □

Lemma 2. [16]

a) *For any function $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_q \leq B_{q,p}\|\nabla u\|_p$, for all $1 \leq q \leq p^*$ where $p^* = \frac{np}{n-p}$ if $n > p$ and $p^* = \infty$ if $n \leq p$. The best constant $B_{q,p}$ depends only on Ω, n, p and q. We will denote the constant $B_{p,p}$ by B_p.*
b) *For any $u \in W_0^{1,p}(\Omega)$, $p \geq 1$, and $r \geq 1$, the inequality*

$$\|u\|_q \leq C\|\nabla u\|_p^\theta \|u\|_r^{1-\theta},$$

is valid, where $\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1}$ and for $p \geq n = 1$, $r \leq q \leq \infty$; for $n > 1$ and $p < n$, $q \in [r, p^*]$ if $r < p^*$ and $q \in [p^*, r]$ if $r \geq p^*$ for $p = n > 1$, $r \leq q \leq \infty$; for $p > n > 1$, $r \leq q \leq \infty$.

Here, the constant C depends on n, p, q and r.

Lemma 3. [17] *Assume that $f : R^+ \to R^+$ be a nonincreasing function and σ is a nonnegative constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s)ds \leq \frac{1}{\omega}f^\sigma(0)f(t), \quad \forall t \geq 0.$$

Hence

(a) $f(t) \leq f(0)e^{1-\omega t}$, $\forall t \geq 0$, *whenever* $\sigma = 0$;

(b) $f(t) \leq f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, $\forall t \geq 0$, *whenever* $\sigma > 0$.

3. Main results

Now as in [18], we introduce the follows sets:

$$
\begin{aligned}
\mathcal{W}_1 &= \{u \in H_0^1(\Omega)\backslash\{0\} : J(u) < d\},\\
\mathcal{W}_2 &= \{u \in H_0^1(\Omega)\backslash\{0\} : J(u) = d\},\\
\mathcal{W} &= \mathcal{W}_1 \cup \mathcal{W}_2,\\
\mathcal{W}_1^+ &= \{u \in \mathcal{W}_1 : I(u) > 0\},\\
\mathcal{W}_2^+ &= \{u \in \mathcal{W}_2 : I(u) > 0\},\\
\mathcal{W}^+ &= \mathcal{W}_1^+ \cup \mathcal{W}_2^+,\\
\mathcal{W}_1^- &= \{u \in \mathcal{W}_1 : I(u) < 0\},\\
\mathcal{W}_2^- &= \{u \in \mathcal{W}_2 : I(u) < 0\},\\
\mathcal{W}^- &= \mathcal{W}_1^- \cup \mathcal{W}_2^-.
\end{aligned}
$$

Definition 4. (Maximal existence time). Assume that u be weak solutions of problem (1). We define the maximal existence time $T_{\max}$ as follows

$$T_{\max} = \sup\{T > 0 : u(t) \text{ exists on } [0, T]\}.$$

Then

(i) If $T_{\max} < \infty$, we see that u blows up in finite time and $T_{\max}$ is the blow-up time;
(ii) If $T_{\max} = \infty$, we see that u is global.

Definition 5. (Weak solution). We define a function $u \in L^\infty(0, T; H_0^1(\Omega))$ with $u_t \in L^2(0, T; H_0^1(\Omega))$ to be a weak solution of problem (1) over $[0, T]$, if it satisfies the initial condition $u(0) = u_0 \in H_0^1(\Omega)\backslash\{0\}$, and

$$
\begin{aligned}
&< u_t, w > + < \nabla u, \nabla w > + < \|\nabla u\|^{2\gamma} \nabla u, \nabla w > + < \nabla u_t, \nabla w >\\
&= \int_\Omega u^{k-1} \ln|u|\, w dx - \int_\Omega \oint_\Omega u^{k-1} \ln|u|\, w ds dx,
\end{aligned}
$$

for all $w \in H_0^1(\Omega)$, and for a.e. $t \in [0, T]$.

Theorem 6. *(Global existence). Let $u_0 \in \mathcal{W}^+$, $0 < J(u_0) < d$ and $I(u) > 0$. Hence, there is a unique global weak solution u of (1) satisfying $u(0) = u_0$. We obtain $u(t) \in \mathcal{W}^+$ holds for all $0 \leq t < +\infty$, and the energy estimate*

$$\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + J(u(t)) \leq J(u_0), \quad 0 \leq t \leq +\infty.$$

Also, the solution decay exponentially provided $u_0 \in \mathcal{W}_1^+$.

Proof. (Global existence) In the space $H_0^1(\Omega)$, we take a Galerkin bases $\{w_j\}_{j=1}^\infty$ and define the finite dimensional space

$$V_m = span\{w_1, w_2, ..., w_m\}.$$

Let u_{0m} be an element of V_m such that

$$u_{0m} = \sum_{j=1}^m a_{mj} w_j \to u_0 \quad \text{strongly in } H_0^1(\Omega), \tag{9}$$

as $m \to \infty$. We can find the approximate solution $u_m(x,t)$ of the problem (1) in the form

$$u_m(x,t) = \sum_{j=1}^{m} a_{mj}(t) w_j(x), \tag{10}$$

where the coefficients a_{mj} $(1 \le j \le m)$ satisfy the ordinary differential equations

$$\begin{aligned} &\int_\Omega u_{mt} w_i dx + \int_\Omega \nabla u_m \nabla w_i dx + \int_\Omega \|\nabla u_m\|^{2\gamma} \nabla u_m \nabla w_i dx + \int_\Omega \nabla u_{mt} \nabla w_i dx \\ &= \int_\Omega |u_m|^{k-1} \ln |u_m| \, w_i dx - \int_\Omega \oint_\Omega |u_m|^{k-1} \ln |u_m|) w_i ds dx, \end{aligned} \tag{11}$$

for $i \in \{1, 2, ..., m\}$, with the initial condition

$$a_{mj}(0) = a_{mj}, \quad j \in \{1, 2, ..., m\}. \tag{12}$$

Now, multiplying (11) by a'_{mi}, summing over i from 1 to m and integrating with related to time variable on $[0,t]$, we obtain

$$\int_0^t \|u_{ms}(s)\|^2_{H_0^1(\Omega)} ds + J(u_m(t)) \le J(u_{0m}), \quad 0 \le t \le T_{\max}, \tag{13}$$

where $T_{\max}$ is the maximal existence time of solution $u_m(t)$. By (9), (13) and the continuity of J that

$$J(u_m(0)) \to J(u_{0m}), \text{ as } m \to \infty, \tag{14}$$

with $J(u_0) < d$ and the continuity of functional J, by (14), we have

$$J(u_{0m}) < d, \quad \text{for sufficiently large } m.$$

And therefore, from (13), we obtain

$$\int_0^t \|u_{ms}(s)\|^2_{H_0^1(\Omega)} ds + J(u_m(t)) < d, \quad 0 \le t \le T_{\max}, \tag{15}$$

for sufficiently large m. Next, we will show that

$$u_m(t) \in \mathcal{W}_1^+, \quad t \in [0, T_{\max}), \tag{16}$$

for sufficiently large m. We suppose that (16) does not hold and think that there exists a smallest time t_0 such that $u_m(t_0) \notin \mathcal{W}_1^+$. Then, by continuity of $u_m(t_0) \in \partial\mathcal{W}_1^+$. So, we obtain

$$J(u_m(t_0)) = d, \tag{17}$$

and

$$I(u_m(t_0)) = 0. \tag{18}$$

It is clear that (17) could not occur by (15) while if (18) holds then, by definition of d, we get

$$J(u_m(t_0)) \ge \inf_{u \in \mathcal{N}} J(u) = d,$$

which contradicts with (15). Thus, we get (16), i.e., $J(u_m(t)) < d$ and $I(u_m(t)) > 0$, for any $t \in [0, T_{\max})$, for sufficiently large m. Hence, by (6), we get

$$\begin{aligned} d &> J(u_m(t)) \\ &= \frac{1}{k} I(u_m(t)) + \frac{k-2}{2k} \|\nabla u_m(t)\|^2 + \frac{k-2\gamma-2}{2k(\gamma+1)} \|\nabla u_m(t)\|^{2(\gamma+1)} + \frac{1}{k^2} \int_\Omega |u|^k dx \\ &\ge \frac{k-2}{2k} \|\nabla u_m(t)\|^2 + \frac{\gamma}{2(\gamma+1)} \|\nabla u_m(t)\|^{2(\gamma+1)} + \frac{1}{k^2} \int_\Omega |u|^k dx \ge \frac{k-2}{2k} \|\nabla u_m(t)\|^2 . \end{aligned}$$

And therefore, we deduce from (15) that

$$\int_0^t \|u_{ms}(s)\|^2_{H_0^1(\Omega)}\, ds + \frac{k-2}{2k}\|\nabla u_m(t)\|^2 < d.$$

This inequality implies $T_{\max} = +\infty$. Further, by the logarithmic inequality, we get

$$\begin{aligned}\|\nabla u_m(t)\|^2 =& 2J(u_m(t)) + \frac{2}{k}\int_\Omega |u_m(t)|^k \ln|u_m(t)|\,dx - \frac{2}{k^2}\|u_m(t)\|^2 - \frac{1}{(\gamma+1)}\|\nabla u_m(t)\|^{2(\gamma+1)}\\ \le& 2J(u_m(0)) + \frac{2}{k}\int_\Omega |u_m(t)|^k \ln|u_m(t)|\,dx.\end{aligned}$$

This implies that

$$\|\nabla u_m(t)\|^2 \le 2J(u_m(0)) + \frac{2}{k}\int_\Omega |u_m(t)|^k \ln|u_m(t)|\,dx.$$

We deduce that

$$\|\nabla u_m(t)\|^2 \le C_d, \quad \forall t \in [0, T_{\max}).$$

(Decay estimates) Thanks to $u_0 \in \mathcal{W}_1^+$, we deduce from (6) that

$$\begin{aligned}J(u_0) &> J(u(t))\\ &= \frac{1}{k}I(u(t)) + \frac{k-2}{2k}\|\nabla u(t)\|^2 + \frac{k-2\gamma-2}{2k(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)} + \frac{1}{k^2}\int_\Omega |u|^k\,dx,\\ &\ge \frac{k-2}{2k}\|\nabla u(t)\|^2 + \frac{k-2\gamma-2}{2k(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)} + \frac{1}{k^2}\int_\Omega |u|^k\,dx.\end{aligned} \tag{19}$$

By $I(u(t)) > 0$, (7) and Lemma 1, there exists a $\lambda^* > 1$ such that $I(\lambda^* u(t)) = 0$. We have

$$\begin{aligned}d &\le J(\lambda^* u(t))\\ &= \frac{1}{k}I(\lambda^* u(t)) + \frac{k-2}{2k}\|\nabla(\lambda^* u(t))\|^2 + \frac{k-2\gamma-2}{2k(\gamma+1)}\|\nabla(\lambda^* u(t))\|^{2(\gamma+1)} + \frac{1}{k^2}\int_\Omega |\lambda^* u(t)|^k\,dx\\ &= \frac{k-2}{2k}\|\nabla(\lambda^* u(t))\|^2 + \frac{k-2\gamma-2}{2k(\gamma+1)}\|\nabla(\lambda^* u(t))\|^{2(\gamma+1)} + \frac{1}{k^2}\int_\Omega |\lambda^* u(t)|^k\,dx\\ &= (\lambda^*)^2\left(\frac{k-2}{2k}\right)\|\nabla u(t)\|^2 + (\lambda^*)^{2(\gamma+1)}\left(\frac{k-2\gamma-2}{2k(\gamma+1)}\right)\|\nabla u(t)\|^{2(\gamma+1)} + (\lambda^*)^k\left(\frac{1}{k^2}\right)\int_\Omega |u(t)|^k\,dx\\ &\le (\lambda^*)^k\left(\frac{k-2}{2k}\|\nabla u(t)\|^2 + \frac{k-2\gamma-2}{2k(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)} + \frac{1}{k^2}\int_\Omega |u(t)|^k\,dx\right).\end{aligned} \tag{20}$$

Using (19) and (20), we have $d \le (\lambda^*)^k J(u_0)$, which implies that

$$\lambda^* \ge \left(\frac{d}{J(u_0)}\right)^{\frac{1}{k}}. \tag{21}$$

By (5), we get

$$\begin{aligned}0 =& I(\lambda^* u(t)) = \|\nabla(\lambda^* u(t))\|^2 + \|\nabla(\lambda^* u(t))\|^{2(\gamma+1)} - \int_\Omega |\lambda^* u(t)|^k \ln|\lambda^* u(t)|\,dx\\ =& (\lambda^*)^2\|\nabla u(t)\|^2 + (\lambda^*)^{2(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)} - (\lambda^*)^k\int_\Omega |u(t)|^k\ln|u(t)|\,dx - (\lambda^*)^k\ln(\lambda^*)\int_\Omega |u(t)|^k\,dx\\ =& (\lambda^*)^k I(u(t)) + (\lambda^*)^2\|\nabla u(t)\|^2 + (\lambda^*)^{2(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)}\\ &- (\lambda^*)^k\|\nabla u(t)\|^2 - (\lambda^*)^k\|\nabla u(t)\|^{2(\gamma+1)} - (\lambda^*)^k\ln(\lambda^*)\int_\Omega |u(t)|^k\,dx\\ =& (\lambda^*)^k I(u(t)) + \left[(\lambda^*)^2 - (\lambda^*)^k\right]\|\nabla u(t)\|^2\\ &+ \left[(\lambda^*)^{2(\gamma+1)} - (\lambda^*)^k\right]\|\nabla u(t)\|^{2(\gamma+1)} - (\lambda^*)^k\ln(\lambda^*)\int_\Omega |u(t)|^k\,dx.\end{aligned} \tag{22}$$

Using (21) and (22), we have

$$\begin{aligned}(\lambda^*)^k I(u(t)) &= \left[(\lambda^*)^k - (\lambda^*)^2\right] \|\nabla u(t)\|^2 + \left[(\lambda^*)^k - (\lambda^*)^{2(\gamma+1)}\right] \|\nabla u(t)\|^{2(\gamma+1)} + (\lambda^*)^k \ln(\lambda^*) \int_\Omega |u\,(t)|^k\,dx \\ &\geq \left[(\lambda^*)^k - (\lambda^*)^2\right] \int_\Omega |u\,(t)|^k\,dx,\end{aligned}$$

which implies that

$$I(u(t)) \geq \left[1 - (\lambda^*)^{2-k}\right] \|\nabla u(t)\|^2. \tag{23}$$

It follows from (21) and (23) that

$$\begin{aligned}I(u(t)) &\geq \left[1 - (\lambda^*)^{2-k}\right] \|\nabla u(t)\|^2 \\ &\geq \left[1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-k}{k}}\right] \|\nabla u(t)\|^2 \\ &\geq C\left[1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-k}{k}}\right] \|u(t)\|^2,\end{aligned} \tag{24}$$

where C is constant. Hence, by (24), we get

$$\begin{aligned}I(u(t)) &\geq \frac{1}{2}\left[1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-k}{k}}\right] \|\nabla u(t)\|^2 + \frac{C}{2}\left[1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-k}{k}}\right] \|u(t)\|^2 \\ &\geq C_1\left(\|\nabla u(t)\|^2 + \|u(t)\|^2\right) \\ &= C_1 \|u(t)\|^2_{H_0^1(\Omega)},\end{aligned} \tag{25}$$

where

$$C_1 = \max\left\{\frac{1}{2}\left[1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-k}{k}}\right], \frac{C}{2}\left[1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-k}{k}}\right]\right\}.$$

Integrating the $I(u(s))$ with respect to s over (t, T), we get

$$\begin{aligned}\int_t^T I(u(s))ds &= -\int_t^T \int_\Omega u_s(s)u(s)dxds - \int_t^T \int_\Omega \nabla u_s(s)\nabla u(s)dxds \\ &= \frac{1}{2}\|u(t)\|^2_{H_0^1(\Omega)} - \frac{1}{2}\|u(T)\|^2_{H_0^1(\Omega)} \\ &\leq \frac{1}{2}\|u(t)\|^2_{H_0^1(\Omega)}.\end{aligned} \tag{26}$$

From (25) and (26), we have

$$\int_t^T C_1 \|u(t)\|^2_{H_0^1(\Omega)}\,ds \leq \frac{1}{2}\|u(t)\|^2_{H_0^1(\Omega)} \text{ for all } t \in [0, T]. \tag{27}$$

Let $T \to +\infty$ in (27), we can have

$$\int_t^\infty \|u(t)\|^2_{H_0^1(\Omega)}\,ds \leq \frac{1}{2C_1}\|u(t)\|^2_{H_0^1(\Omega)}.$$

From Lemma 3, we get

$$\|u(t)\|^2_{H_0^1(\Omega)} \leq \|u(0)\|^2_{H_0^1(\Omega)}\,e^{1-2C_1 t}.$$

The above inequality satisfies that the solution u decays exponentially. □

Theorem 7. *(Blow up). Let $u_0 \in W_1^-$ and assume that $u(t)$ be a unique weak solution to the problem (1). Then $u(t)$ blows up in finite time, that is, there exists $T_* > 0$ such that*

$$\lim_{t\to T_*} \|u(t)\|^2_{H_0^1(\Omega)} = \infty.$$

Proof. We show that $u(t)$ blows up at a finite time. Using contradiction, we suppose that $u(t)$ is global. We contract a function $\phi : [0,\infty) \to \mathbb{R}^+$, and

$$\phi(t) = \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, ds. \tag{28}$$

Then, thorough direct calculation, we have

$$\phi'(t) = \|u(s)\|^2_{H_0^1(\Omega)} = 2\int_0^t \int_\Omega (u_s u + \nabla u_s \nabla u)\, dxds. \tag{29}$$

By (5) and (29), we have

$$\begin{aligned}
\phi''(t) &= 2\int_\Omega (u_s u + \nabla u_s \nabla u)\, dx \\
&= 2\int_\Omega u\,(u_s - \Delta u_s)\, dx \\
&= 2\int_\Omega |u|^k \ln|u|\, dx + 2\int_\Omega M(\|\nabla u\|^2) u \Delta u dx \\
&= 2\int_\Omega |u|^k \ln|u|\, dx - 2\int_\Omega (1+\|\nabla u\|^{2\gamma})(\nabla u)^2 dx \\
&= -2I(u).
\end{aligned} \tag{30}$$

By (30) and $I(u) < 0$, we get $\phi''(t) > 0$, hence

$$\phi'(t) > \phi'(0) = \|u_0\|^2_{H_0^1(\Omega)} > 0. \tag{31}$$

From the Hölder inequality and combining (30), we get

$$\begin{aligned}
\frac{1}{4}\left(\phi'(t) - \phi'(0)\right)^2 &= \frac{1}{4}\left(\int_0^t \phi''(s) ds\right)^2 \\
&= \left(\int_0^t \int_\Omega (u_s u + \nabla u_s \nabla u)\, dxds\right)^2 \\
&\leq \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, ds \int_0^t \|u_s\|^2_{H_0^1(\Omega)}\, ds.
\end{aligned} \tag{32}$$

It follows from (6) and (30) that

$$\begin{aligned}
\phi''(t) = -2I(u) &= -2kJ(u) + (k-2)\|\nabla u\|^2 + \frac{k-2\gamma-2}{\gamma+1}\|\nabla u\|^{2(\gamma+1)} + \frac{2}{k}\|u\|_k^k \\
&\geq -2kJ(u_0) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\, ds + (k-2)\|\nabla u\|^2 + \frac{k-2\gamma-2}{\gamma+1}\|\nabla u\|^{2(\gamma+1)} + \frac{2}{k}\|u\|_k^k \\
&\geq 2k\,(d - J(u_0)) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\, ds.
\end{aligned} \tag{33}$$

Using (28), (32) and (33), we get

$$\begin{aligned}
\phi(t)\phi''(t) &\geq 2k\int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, ds \int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\, ds + 2k\,(d - J(u_0))\,\phi(t) \\
&\geq 2k\,(d - J(u_0))\,\phi(t) + \frac{k}{2}\left(\phi'(t) - \phi'(0)\right)^2.
\end{aligned} \tag{34}$$

By (34), we get

$$\phi(t)\phi''(t) - \frac{k}{2}\left(\phi'(t) - \phi'(0)\right)^2 \geq 2k(d - J(u_0))\,\|u_0\|^2_{H_0^1(\Omega)}\, t_0 > 0. \tag{35}$$

Choose $T > t_0$ sufficiently large and let

$$\psi(t) = \phi(t) + (T - t)\,\|u_0\|^2_{H_0^1(\Omega)}\, t_0,\ \forall t \in [0, T].$$

Hence, $\mu(t) \geq \phi(t) > 0$, $\mu'(t) = \phi'(t) - \phi'(0)$ and $\mu''(t) = \phi''(t) > 0$, so (35) implies

$$\mu(t)\mu''(t) - \frac{k}{2}\mu'(t)^2 \geq 2k\,(d - J(u_0))\,\|u_0\|^2_{H_0^1(\Omega)}\, t_0 > 0,\ \text{for all } t \in [t_0, T]. \tag{36}$$

Let $\psi(t) = \mu(t)^{-\frac{k-2}{2}}$. Thus,

$$\psi'(t) = -\frac{k-2}{2}\mu(t)^{-\frac{k}{2}}\mu'(t). \tag{37}$$

From (36) and (37), we get

$$\psi''(t) = \frac{k-2}{2}\mu(t)^{-\frac{k+2}{2}}\left[\frac{k}{2}\mu'(t)^2 - \mu(t)\mu''(t)\right] < 0,\ \text{for all } t \in [t_0, T].$$

Therefore, for any sufficiently large $T > t_0$, $\psi(t)$ is a concave function in $[t_0, T]$. Since $\psi(t_0) > 0$ and $\psi'(t_0) < 0$, there exists a finite time T_* such that

$$\lim_{t \to T_*^-} \psi(t) = 0.$$

Consequently,

$$\lim_{t \to T_*^-} \mu(t) = \infty,$$

which satisfies

$$\lim_{t \to T_*^-} \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, ds = \infty,$$

therefore, we have

$$\lim_{t \to T_*^-} \|u(s)\|^2_{H_0^1(\Omega)} = \infty.$$

This contradicts with $u(t)$ being a global solution. □

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