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Degree-based topological indices of product graphs

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Abstract: In this paper, we obtain the quantitative calculation formula of the degree-based topological indices of four standard product for the path and regular graphs, which unify to solve the question on product of these basic graphs without having to deal with it one by one separately. As applications, we give corresponding calculation formula of the general Randić index, the first general Zagreb index and the general sum-connectivity index.

Keywords: Degree-based topological indices; Cartesian product; Direct product; Strong product; Lexicographic product.

MSC: 05C07; 05C09; 05C76; 05C90; 05C92.

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1. Introduction

Throughout the article, G is a simple undirected connected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G is called order and size, respectively. If the vertices u and v are adjacent, then we write $u \sim v$. For $v \in V(G)$, $d_v = d_G(v)$ denotes the degree of vertex v in G. Denote by P_n and $K_{1,n-1}$ the path and star with n vertices, respectively.

Cheminformatics is a new interdiscipline composed of chemistry, mathematics and information science, which contributes a major role in the field of chemical sciences by implementing graph theory to mathematical modeling of chemical occurrence. In cheminformatics, the topological indices play a significant role in predicting the biological activities and properties of chemical compounds due to the fact that the numerical characteristics of topological indices reflect certain physico-chemical properties of chemical compounds, such as boiling point, stability, strain energy etc. A large number of topological indices have been studied in the models of Quantitative structure-activity relationships (QSAR) and structure-property relationships (QSPR), such as Wiener index, Randić index, Zagreb index, ABC index and so on.

The study on degree-based topological indices has been one of the hotspots in cheminformatics [1]. Let $K = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \le i \le j \le n-1\}$ and $m_{i,j} = m_{i,j}(G)$ be the number of edges in G joining vertices of degree i and j. For any set of numbers $\{\varphi_{i,j}\}_{(i,j)\in K}$, the general formula of degree-based topological indices is

$$DTI(G) = \sum_{(i,j)\in K} m_{i,j}(G)\, \varphi_{i,j}.$$

In particular, we obtain the first Zagreb index and the second Zagreb index when $\varphi_{i,j} = i + j$ and $\varphi_{i,j} = ij$, respectively.

In 1998, the general Randić index of a graph G, introduced by Bollobás and Erdős [2], is defined as

$$R^t = R^t(G) = \sum_{v_i v_j \in E(G)} \left(d_i d_j\right)^t, \quad t \in \mathbb{R}.$$

Clearly, we have that R^0 is the number of edges, $R^{-\frac{1}{2}}$ is the Randić index [3], R^{-1} is the modified second Zagreb index [3], $R^{\frac{1}{2}}$ is the reciprocal Randić index [4], R^2 is the second Hyper-Zagreb index [4], R^1 is the second Zagreb index [5], etc.

In 2005, the first general Zagreb index of a graph G was introduced by Li and Zheng [6] and is defined as

$$Z^t = Z^t(G) = \sum_{v_i \in V(G)} d_i^t = \sum_{v_i v_j \in E(G)} \left(d_i^{t-1} + d_j^{t-1}\right), \quad t \in \mathbb{R}.$$

It is easy to see that Z^0 is the number of vertices, Z^1 is twice the number of edges, Z^2 is the first Zagreb index [5], Z^3 is the forgotten topological index [7], etc.

In 2010, Zhou and Trinajstić [8] proposed the general sum-connectivity index of a graph G as follows:

$$\chi^t = \chi^t(G) = \sum_{v_i v_j \in E(G)} \left(d_i + d_j\right)^t, \quad t \in \mathbb{R}.$$

It is not difficult to find that $2\chi^{-1}$ is the harmonic index [9], $\chi^{-\frac{1}{2}}$ is the sum-connectivity index [10], $\chi^{\frac{1}{2}}$ is the reciprocal sum-connectivity index [11], etc.

The product graphs are useful in constructing many important structural models with regularities [12], especially the following four standard product graphs which are widely used in network design [13], multiprocessor system [14], automata theory [15] and other fields. Let G_1 and G_2 be two graphs with disjoint vertex sets $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$, respectively. The Cartesian product of G_1 and G_2, denoted by $G_1 \square G_2$ is the graph, where $(u_i, v_j) \sim (u_r, v_s)$ if either ($u_i = u_r$ and $v_j \sim v_s$ in G_2) or ($u_i \sim u_r$ in G_1 and $v_j = v_s$). The direct product or Kronecker product of G_1 and G_2, denoted by $G_1 \otimes G_2$, is the graph where $(u_i, v_j) \sim (u_r, v_s)$ if $u_i \sim u_r$ in G_1 and $v_j \sim v_s$ in G_2. The strong product of G_1 and G_2, denoted by $G_1 \boxtimes G_2$, is graph where $(u_i, u_j) \sim (u_r, u_s)$ if either ($u_i = u_r$ and $u_j \sim u_s$ in G_2) or ($u_i \sim u_r$ in G_1 and $u_j = u_s$) or ($u_i \sim u_r$ in G_1 and $u_j \sim u_s$ in G_2). The lexicographic product of G_1 and G_2, denoted by $G_1[G_2]$, is the graph where $(u_i, v_j) \sim (u_r, v_s)$ if either ($u_i \sim u_r$ in G_1) or ($u_i = u_r$ and $v_j \sim v_s$ in G_2).

In this paper, we give a unified approach to solve the computational problems of degree-based topological indices of standard product graphs for the path and regular graphs, which is generalization of many specific degree-based topological indices. As applications, the corresponding calculation formulas of the general Randić index, the first general Zagreb index and the general sum-connectivity index are obtained.

2. Cartesian product

Theorem 1. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$DTI\left(P_{n_1} \square P_{n_2}\right) = 8\varphi_{2,3} + 2\left(n_1 + n_2 - 6\right)\varphi_{3,3} + 2\left(n_1 + n_2 - 4\right)\varphi_{3,4} + \left(2n_1n_2 - 5n_1 - 5n_2 + 12\right)\varphi_{4,4}$$

for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of Cartesian product, we obtain the basic information on $P_{n_1} \square P_{n_2}$ in the Table 1.

Table 1. The basic information on $P_{n_1} \square P_{n_2}$.

$m_{2,3}$	$m_{3,3}$	$m_{3,4}$	$m_{4,4}$
8	$2(n_1+n_2-6)$	$2(n_1+n_2-4)$	$2n_1n_2 - 5n_1 - 5n_2 + 12$

Thus we have

$$DTI\left(P_{n_1} \square P_{n_2}\right) = \sum_{(i,j)\in K} m_{i,j}(G)\,\varphi_{i,j} = 8\varphi_{2,3} + 2\left(n_1+n_2-6\right)\varphi_{3,3} + 2\left(n_1+n_2-4\right)\varphi_{3,4} + \left(2n_1n_2 - 5n_1 - 5n_2 + 12\right)\varphi_{4,4}.$$

This completes the proof. □

Corollary 1. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$R^t\left(P_{n_1} \square P_{n_2}\right) = 8\cdot 6^t + 2\cdot 9^t\left(n_1+n_2-6\right) + 2\cdot 12^t\left(n_1+n_2-4\right) + 16^t\left(2n_1n_2 - 5n_1 - 5n_2 + 12\right),$$
$$Z^t\left(P_{n_1} \square P_{n_2}\right) = 8\left(2^{t-1} + 3^{t-1}\right) + 4\cdot 3^{t-1}\left(n_1+n_2-6\right) + 2\left(n_1+n_2-4\right)\left(3^{t-1} + 4^{t-1}\right) + 2\cdot 4^{t-1}\left(2n_1n_2 - 5n_1 - 5n_2 + 12\right),$$

$$\chi^t\left(P_{n_1}\,\square\,P_{n_2}\right)=8\cdot 5^t+2\cdot 6^t\left(n_1+n_2-6\right)+2\cdot 7^t\left(n_1+n_2-4\right)+8^t\left(2n_1n_2-5n_1-5n_2+12\right)$$

for $n_1\geq n_2\geq 3$.

Theorem 2. *Let P_{n_1} and G_r be a path and a r-regular graph of order n_1 and n_2, respectively. Then*

$$DTI\left(P_{n_1}\,\square\,G_r\right)=rn_2\varphi_{r+1,r+1}+2n_2\varphi_{r+1,r+2}+\frac{1}{2}[rn_2\left(n_1-2\right)+2n_1n_2-6n_2]\varphi_{r+2,r+2}$$

for $n_1\geq n_2\geq 2$.

Proof. By the definition of Cartesian product, we obtain the basic information on $P_{n_1}\,\square\,G_r$ in the following Table 2.

Table 2. The basic information on $P_{n_1}\,\square\,G_r$.

$m_{r+1,r+1}$	$m_{r+1,r+2}$	$m_{r+2,r+2}$
rn_2	$2n_2$	$\frac{rn_2\left(n_1-2\right)}{2}+n_1n_2-3n_2$

Thus we have

$$DTI\left(P_{n_1}\,\square\,G_r\right)=\sum_{(i,j)\in K}m_{i,j}\left(G\right)\varphi_{i,j}=rn_2\varphi_{r+1,r+1}+2n_2\varphi_{r+1,r+2}+\frac{1}{2}[rn_2\left(n_1-2\right)+2n_1n_2-6n_2]\varphi_{r+2,r+2}.$$

This completes the proof. □

Corollary 2. *Let P_{n_1} and G_r be a path and a r-regular graph of order n_1 and n_2, respectively. Then*

$$R^t\left(P_{n_1}\,\square\,G_r\right)=rn_2\left(r+1\right)^{2t}+2n_2\left(r+1\right)^t\left(r+2\right)^t+\left[\frac{rn_2\left(n_1-2\right)}{2}+n_2\left(n_1-3\right)\right]\left(r+2\right)^{2t},$$

$$Z^t\left(P_{n_1}\,\square\,G_r\right)=2rn_2\left(r+1\right)^{t-1}+2n_2\left[\left(r+1\right)^{t-1}+\left(r+2\right)^{t-1}\right]+2\left(r+2\right)^{t-1}\left[\frac{rn_2\left(n_1-2\right)}{2}+n_2\left(n_1-3\right)\right],$$

$$\chi^t\left(P_{n_1}\,\square\,G_r\right)=2^trn_2\left(r+1\right)^t+2n_2\left(2r+3\right)^t+2^t\left(r+2\right)^t\left[\frac{rn_2\left(n_1-2\right)}{2}+n_2\left(n_1-3\right)\right]$$

for $n_1\geq n_2\geq 2$.

Theorem 3. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$DTI\left(G_r\,\square\,P_{n_2}\right)\;=\;rn_1\varphi_{r+1,r+1}+2n_1\varphi_{r+1,r+2}+\frac{1}{2}\left[rn_1\left(n_2-2\right)+2n_1n_2-6n_1\right]\varphi_{r+2,r+2}$$

for $n_1\geq n_2\geq 2$.

Proof. By the definition of Cartesian product, we obtain the basic information on $G_r\,\square\,P_{n_2}$ in the following Table 3.

Table 3. The basic information on $G_r\,\square\,P_{n_2}$.

$m_{r+1,r+1}$	$m_{r+1,r+2}$	$m_{r+2,r+2}$
rn_1	$2n_1$	$\frac{rn_1\left(n_2-2\right)}{2}+n_1n_2-3n_1$

Thus we have

$$DTI\left(G_r\,\square\,P_{n_2}\right)=\sum_{(i,j)\in K}m_{i,j}\left(G\right)\varphi_{i,j}=rn_1\varphi_{r+1,r+1}+2n_1\varphi_{r+1,r+2}+\frac{1}{2}\left[rn_1\left(n_2-2\right)+2n_1n_2-6n_1\right]\varphi_{r+2,r+2}.$$

This completes the proof. □

Corollary 3. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$R^t\left(G_r \square P_{n_2}\right) = rn_1\left(r+1\right)^{2t} + 2n_1\left(r+1\right)^t\left(r+2\right)^t + \left[\frac{rn_1\left(n_2-2\right)}{2} + n_1\left(n_2-3\right)\right]\left(r+2\right)^{2t},$$
$$Z^t\left(G_r \square P_{n_2}\right) = 2rn_1\left(r+1\right)^{t-1} + 2n_1\left[\left(r+1\right)^{t-1} + \left(r+2\right)^{t-1}\right] + 2\left(r+2\right)^{t-1}\left[\frac{rn_1\left(n_2-2\right)}{2} + n_1\left(n_2-3\right)\right],$$
$$\chi^t\left(G_r \square P_{n_2}\right) = 2^t rn_1\left(r+1\right)^t + 2n_1\left(2r+3\right)^t + 2^t\left(r+2\right)^t\left[\frac{rn_1\left(n_2-2\right)}{2} + n_1\left(n_2-3\right)\right]$$

for $n_1 \geq n_2 \geq 2$.

Theorem 4. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$DTI\left(G_1 \square G_2\right) = \frac{n_1 n_2\left(r_1+r_2\right)}{2}\varphi_{r_1+r_2, r_1+r_2}$$

for $n_1 \geq n_2 \geq 2$.

Proof. By the definition of Cartesian product, we have $G_1 \square G_2$ is a (r_1+r_2)-regular graph with $\frac{n_1n_2(r_1+r_2)}{2}$ edges. Thus

$$DTI\left(G_1 \square G_2\right) = \sum_{(i,j)\in K} m_{i,j}\left(G\right)\varphi_{i,j} = \frac{n_1 n_2\left(r_1+r_2\right)}{2}\varphi_{r_1+r_2, r_1+r_2}.$$

This completes the proof. □

Corollary 4. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$R^t\left(G_1 \square G_2\right) = \frac{n_1 n_2\left(r_1+r_2\right)^{2t+1}}{2},$$
$$Z^t\left(G_1 \square G_2\right) = n_1 n_2\left(r_1+r_2\right)^t,$$
$$\chi^t\left(G_1 \square G_2\right) = 2^{t-1} n_1 n_2\left(r_1+r_2\right)^{t+1}$$

for $n_1 \geq n_2 \geq 2$.

3. Direct product

Theorem 5. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$DTI\left(P_{n_1} \otimes P_{n_2}\right) = 4\varphi_{1,4} + 4\varphi_{2,2} + 4\left(n_1+n_2-6\right)\varphi_{2,4} + 2\left(n_1-3\right)\left(n_2-3\right)\varphi_{4,4}$$

for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of direct product, we obtain the basic information on $P_{n_1} \otimes P_{n_2}$ in the following Table 4.

Table 4. The basic information on $P_{n_1} \otimes P_{n_2}$.

$m_{1,4}$	$m_{2,2}$	$m_{2,4}$	$m_{4,4}$
4	4	$4\left(n_1+n_2-6\right)$	$2\left(n_1-3\right)\left(n_2-3\right)$

Thus we have

$$DTI\left(P_{n_1} \otimes P_{n_2}\right) = \sum_{(i,j)\in K} m_{i,j}\left(G\right)\varphi_{i,j} = 4\varphi_{1,4} + 4\varphi_{2,2} + 4\left(n_1+n_2-6\right)\varphi_{2,4} + 2\left(n_1-3\right)\left(n_2-3\right)\varphi_{4,4}.$$

This completes the proof. □

Corollary 5. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$\begin{aligned}
R^t\left(P_{n_1}\otimes P_{n_2}\right) &=8\cdot 4^t+2\cdot 8^t\left(n_1n_2-n_1-n_2-3\right),\\
Z^t\left(P_{n_1}\otimes P_{n_2}\right) &=4\left[1+2^t+\left(n_1+n_2-6\right)\left(2^{t-1}+4^{t-1}\right)\right]+4^t\left[1+\left(n_1-3\right)\left(n_2-3\right)\right],\\
\chi^t\left(P_{n_1}\otimes P_{n_2}\right) &=4\left[4^t+5^t+6^t\left(n_1+n_2-6\right)\right]+2^{3t+1}\left(n_1-3\right)\left(n_2-3\right)
\end{aligned}$$

for $n_1\geq n_2\geq 3$.

Theorem 6. *Let P_{n_1} and G_r be a path and a r-regular of order n_1 and n_2, respectively. Then*

$$DTI\left(P_{n_1}\otimes G_r\right)=2rn_2\varphi_{r,2r}+rn_2\left(n_1-3\right)\varphi_{2r,2r}$$

for $n_1\geq n_2\geq 3$.

Proof. By the definition of direct product, we obtain the basic information on $P_{n_1}\otimes G_r$ in the following Table 5.

Table 5. The basic information on $P_{n_1}\otimes G_r$.

$m_{r,2r}$	$m_{2r,2r}$
$2rn_2$	$rn_2\left(n_1-3\right)$

Thus we have

$$DTI\left(P_{n_1}\otimes G_r\right)=\sum_{(i,j)\in K} m_{i,j}\left(G\right)\varphi_{i,j}=2rn_2\varphi_{r,2r}+rn_2\left(n_1-3\right)\varphi_{2r,2r}.$$

This completes the proof. □

Corollary 6. *Let P_{n_1} and G_r be a path and a r-regular of order n_1 and n_2, respectively. Then*

$$\begin{aligned}
R^t\left(P_{n_1}\otimes G_r\right) &=2^{t+1}r^{2t+1}n_2+2^{2t}r^{2t+1}n_2\left(n_1-3\right),\\
Z^t\left(P_{n_1}\otimes G_r\right) &=\left(2+2^t\right)r^tn_2+2^tr^tn_2\left(n_1-3\right),\\
\chi^t\left(P_{n_1}\otimes G_r\right) &=2n_2\cdot 3^t\cdot r^{t+1}+4^tr^{t+1}n_2\left(n_1-3\right)
\end{aligned}$$

for $n_1\geq n_2\geq 3$.

Theorem 7. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$DTI\left(G_r\otimes P_{n_2}\right)=2rn_1\varphi_{r,2r}+rn_1\left(n_2-3\right)\varphi_{2r,2r}$$

for $n_1\geq n_2\geq 3$.

Proof. By the definition of direct product, we obtain the basic information on $G_r\otimes P_{n_2}$ in the following Table 6.

Table 6. The basic information on $G_r\otimes P_{n_2}$.

$m_{r,2r}$	$m_{2r,2r}$
$2rn_1$	$rn_1\left(n_2-3\right)$

Thus we have

$$DTI\left(G_r\otimes P_{n_2}\right)=\sum_{(i,j)\in K} m_{i,j}\left(G\right)\varphi_{i,j}=2rn_1\varphi_{r,2r}+rn_1\left(n_2-3\right)\varphi_{2r,2r}.$$

This completes the proof. □

Corollary 7. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(G_r \otimes P_{n_2}\right) =& 2^{t+1} r^{2t+1} n_1 + 2^{2t} r^{2t+1} n_1 \left(n_2 - 3\right), \\ Z^t\left(G_r \otimes P_{n_2}\right) =& r^t n_1 \left(2 + 2^t\right) + 2^t r^t n_1 \left(n_2 - 3\right), \\ \chi^t\left(G_r \otimes P_{n_2}\right) =& 2 \cdot 3^t \cdot r^{t+1} n_1 + 4^t r^{t+1} n_1 \left(n_2 - 3\right) \end{aligned}$$

for $n_1 \geq n_2 \geq 3$.

Theorem 8. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$DTI\left(G_1 \otimes G_2\right) = \frac{r_1 r_2 n_1 n_2}{2} \varphi_{r_1 r_2, r_1 r_2}$$

for $n_1 \geq n_2 \geq 2$.

Proof. By the definition of direct product, we have $G_1 \otimes G_2$ is a $r_1 r_2$-regular graph with $\frac{r_1 r_2 n_1 n_2}{2}$ edges. Thus

$$DTI\left(G_1 \otimes G_2\right) = \sum_{(i,j) \in K} m_{i,j}\left(G\right) \varphi_{i,j} = \frac{r_1 r_2 n_1 n_2}{2} \varphi_{r_1 r_2, r_1 r_2}.$$

This completes the proof. □

Corollary 8. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(G_1 \otimes G_2\right) =& \frac{n_1 n_2 \left(r_1 r_2\right)^{2t+1}}{2}, \\ Z^t\left(G_1 \otimes G_2\right) =& n_1 n_2 \left(r_1 r_2\right)^t, \\ \chi^t\left(G_1 \otimes G_2\right) =& 2^{t-1} n_1 n_2 \left(r_1 r_2\right)^{t+1} \end{aligned}$$

for $n_1 \geq n_2 \geq 2$.

4. Strong product

Theorem 9. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$DTI\left(P_{n_1} \boxtimes P_{n_2}\right) = 8\varphi_{3,5} + 4\varphi_{3,8} + 2\left(n_1 + n_2 - 4\right)\varphi_{5,5} + \left(6n_1 + 6n_2 - 32\right)\varphi_{5,8} + \left[4n_1 n_2 - 11\left(n_1 + n_2\right) + 30\right]\varphi_{8,8}$$

for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of strong product, we obtain the basic information on $P_{n_1} \boxtimes P_{n_2}$ in the following Table 7.

Table 7. The basic information on $P_{n_1} \boxtimes P_{n_2}$.

$m_{3,5}$	$m_{3,8}$	$m_{5,5}$	$m_{5,8}$	$m_{8,8}$
8	4	$2\left(n_1 + n_2\right) - 8$	$6\left(n_1 + n_2\right) - 32$	$4n_1 n_2 - 11\left(n_1 + n_2\right) + 30$

Thus we have

$$\begin{aligned} DTI\left(P_{n_1} \boxtimes P_{n_2}\right) =& \sum_{(i,j) \in K} m_{i,j}\left(G\right) \varphi_{i,j} \\ =& 8\varphi_{3,5} + 4\varphi_{3,8} + 2\left(n_1 + n_2 - 4\right)\varphi_{5,5} + \left(6n_1 + 6n_2 - 32\right)\varphi_{5,8} + \left[4n_1 n_2 - 11\left(n_1 + n_2\right) + 30\right]\varphi_{8,8}. \end{aligned}$$

This completes the proof. □

Corollary 9. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(P_{n_1} \boxtimes P_{n_2}\right) =& 8\cdot 15^t + 4\cdot 24^t + 25^t\cdot[2(n_1+n_2-4)] + 40^t\cdot(6n_1+6n_2-32) \\ &+ 64^t\cdot[4n_1n_2 - 11(n_1+n_2)+30], \\ Z^t\left(P_{n_1} \boxtimes P_{n_2}\right) =& 8\cdot\left(3^{t-1}+5^{t-1}\right) + 4\cdot\left(3^{t-1}+8^{t-1}\right) + 4\cdot 5^{t-1}[(n_1+n_2)-4] \\ &+ (6n_1+6n_2-32)\cdot\left(5^{t-1}+8^{t-1}\right) + 2\cdot 8^{t-1}[4n_1n_2-11(n_1+n_2)+30], \\ \chi^t\left(P_{n_1} \boxtimes P_{n_2}\right) =& 8^{t+1} + 4\cdot 11^t + 10^t\cdot[2(n_1+n_2)-8] + 13^t\cdot(6n_1+6n_2-32) \\ &+ 16^t\cdot[4n_1n_2-11(n_1+n_2)+30] \end{aligned}$$

for $n_1 \ge n_2 \ge 3$.

Theorem 10. *Let P_{n_1} and G_r be a path and a r-regular of order n_1 and n_2, respectively. Then*

$$DTI\left(P_{n_1} \boxtimes G_r\right) = rn_2\varphi_{2r+1,2r+1} + 2(r+1)n_2\varphi_{2r+1,3r+2} + \frac{1}{2}\left[n_1n_2(3r+2) - 2n_2(4r+3)\right]\varphi_{3r+2,3r+2}$$

for $n_1 > n_2 \ge 2$.

Proof. By the definition of strong product, we obtain the basic information on $P_{n_1} \boxtimes G_r$ in the following Table 8.

Table 8. The basic information on $P_{n_1} \boxtimes G_r$.

$m_{2r+1,2r+1}$	$m_{2r+1,3r+2}$	$m_{3r+2,3r+2}$
rn_2	$2(r+1)n_2$	$n_1n_2\left(\frac{3r}{2}+1\right) - n_2(4r+3)$

Thus we have

$$\begin{aligned} DTI\left(P_{n_1} \boxtimes G_r\right) &= \sum_{(i,j)\in K} m_{i,j}(G)\varphi_{i,j} \\ &= rn_2\varphi_{2r+1,2r+1} + 2(r+1)n_2\varphi_{2r+1,3r+2} + \frac{1}{2}\left[n_1n_2(3r+2) - 2n_2(4r+3)\right]\varphi_{3r+2,3r+2}. \end{aligned}$$

This completes the proof. □

Corollary 10. *Let P_{n_1} and G_r be a path and a r-regular of order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(P_{n_1} \boxtimes G_r\right) =& rn_2(2r+1)^{2t} + 2n_2(r+1)(2r+1)^t(3r+2)^t + \left[n_1n_2\left(\frac{3r}{2}+1\right) - n_2(4r+3)\right](3r+2)^{2t}, \\ Z^t\left(P_{n_1} \boxtimes G_r\right) =& 2rn_2(2r+1)^{t-1} + 2n_2(r+1)\left[(2r+1)^{t-1} + (3r+2)^{t-1}\right] + 2(3r+2)^{t-1}\left[n_1n_2\left(\frac{3r}{2}+1\right) - n_2(4r+3)\right], \\ \chi^t\left(P_{n_1} \boxtimes G_r\right) =& 2^t rn_2(2r+1)^t + 2n_2(r+1)(5r+3)^t + 2^t(3r+2)^t\left[n_1n_2\left(\frac{3r}{2}+1\right) - n_2(4r+3)\right] \end{aligned}$$

for $n_1 > n_2 \ge 2$.

Theorem 11. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$DTI\left(G_r \boxtimes P_{n_2}\right) = rn_1\varphi_{2r+1,2r+1} + 2(r+1)n_1\varphi_{2r+1,3r+2} + \frac{1}{2}\left[n_1n_2(3r+2) - 2n_1(4r+3)\right]\varphi_{3r+2,3r+2}$$

for $n_1 \ge n_2 \ge 3$.

Proof. By the definition of strong product, we obtain the basic information on $G_r \boxtimes P_{n_2}$ in the following Table 9.

Table 9. The basic information on $G_r \boxtimes P_{n_2}$.

$m_{2r+1,2r+1}$	$m_{2r+1,3r+2}$	$m_{3r+2,3r+2}$
rn_1	$2(r+1)n_1$	$n_1n_2\left(\frac{3r}{2}+1\right)-n_1(4r+3)$

Thus we have

$$\begin{aligned} DTI\left(G_r \boxtimes P_{n_2}\right) &= \sum_{(i,j)\in K} m_{i,j}(G)\,\varphi_{i,j} \\ &= rn_1\varphi_{2r+1,2r+1} + 2(r+1)n_1\varphi_{2r+1,3r+2} + \frac{1}{2}\left[n_1n_2(3r+2) - 2n_1(4r+3)\right]\varphi_{3r+2,3r+2}. \end{aligned}$$

This completes the proof. □

Corollary 11. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(G_r \boxtimes P_{n_2}\right) &= rn_1(2r+1)^{2t} + 2n_1(r+1)(2r+1)^t(3r+2)^t + (3r+2)^{2t}\left[n_1n_2\left(\frac{3r}{2}+1\right) - n_1(4r+3)\right], \\ Z^t\left(G_r \boxtimes P_{n_2}\right) &= 2rn_1(2r+1)^{t-1} + 2n_1(r+1)\left[(2r+1)^{t-1} + (3r+2)^{t-1}\right] + 2(3r+2)^{t-1}\left[n_1n_2\left(\frac{3r}{2}+1\right) - n_1(4r+3)\right], \\ \chi^t\left(G_r \boxtimes P_{n_2}\right) &= 2^t rn_1(2r+1)^t + 2n_1(r+1)(5r+3)^t + 2^t(3r+2)^t\left[n_1n_2\left(\frac{3r}{2}+1\right) - n_1(4r+3)\right] \end{aligned}$$

for $n_1 \geq n_2 \geq 3$.

Theorem 12. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$DTI(G_1 \boxtimes G_2) = \frac{n_1n_2(r_1r_2+r_1+r_2)}{2}\varphi_{r_1r_2+r_1+r_2,r_1r_2+r_1+r_2}$$

for $n_1 \geq n_2 \geq 2$.

Proof. By the definition of strong product, we have $G_1 \boxtimes G_2$ is a $(r_1r_2+r_1+r_2)$-regular graph with $\frac{n_1n_2(r_1r_2+r_1+r_2)}{2}$ edges. Thus

$$DTI(G_1 \boxtimes G_2) = \sum_{(i,j)\in K} m_{i,j}(G)\,\varphi_{i,j} = \frac{n_1n_2(r_1r_2+r_1+r_2)}{2}\varphi_{r_1r_2+r_1+r_2,r_1r_2+r_1+r_2}.$$

This completes the proof. □

Corollary 12. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t(G_1 \boxtimes G_2) &= \frac{n_1n_2(r_1r_2+r_1+r_2)^{2t+1}}{2}, \\ Z^t(G_1 \boxtimes G_2) &= n_1n_2(r_1r_2+r_1+r_2)^t, \\ \chi^t(G_1 \boxtimes G_2) &= 2^{t-1}n_1n_2(r_1r_2+r_1+r_2)^{t+1} \end{aligned}$$

for $n_1 \geq n_2 \geq 2$.

5. Lexicographic product

Theorem 13. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$\begin{aligned} DTI\left(P_{n_1}[P_{n_2}]\right) = {} & 4\varphi_{n_2+1,n_2+2} + 8\varphi_{n_2+1,2n_2+1} + 4(n_2-2)\varphi_{n_2+1,2n_2+2} + 2(n_2-3)\varphi_{n_2+2,n_2+2} + 4(n_2-2)\varphi_{n_2+2,2n_2+1} \\ & + 2(n_2-2)^2\varphi_{n_2+2,2n_2+2} + 4(n_1-3)\varphi_{2n_2+1,2n_2+1} + [2(n_1-2) + 4(n_1-3)(n_2-2)]\varphi_{2n_2+1,2n_2+2} \\ & + [(n_1-2)(n_2-3) + (n_1-3)(n_2-2)^2]\varphi_{2n_2+2,2n_2+2} \end{aligned}$$

for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of lexicographic product, we obtain the basic information on $P_{n_1}[P_{n_2}]$ in the following Table 10.

Table 10. The basic information on $P_{n_1}[P_{n_2}]$.

m_{n_2+1,n_2+2}	4
$m_{n_2+1,2n_2+1}$	8
$m_{n_2+1,2n_2+2}$	$4(n_2-2)$
m_{n_2+2,n_2+2}	$2(n_2-3)$
$m_{n_2+2,2n_2+1}$	$4(n_2-2)$
$m_{n_2+2,2n_2+2}$	$2(n_2-2)^2$
$m_{2n_2+1,2n_2+1}$	$4(n_1-3)$
$m_{2n_2+1,2n_2+2}$	$2(n_1-2)+4(n_1-3)(n_2-2)$
$m_{2n_2+2,2n_2+2}$	$(n_1-2)(n_2-3)+(n_1-3)(n_2-2)^2$

Thus we have

$$\begin{aligned}
DTI\left(P_{n_1}[P_{n_2}]\right) &= \sum_{(i,j)\in K} m_{i,j}(G)\,\varphi_{i,j} \\
&= 4\varphi_{n_2+1,n_2+2} + 8\varphi_{n_2+1,2n_2+1} + 4(n_2-2)\,\varphi_{n_2+1,2n_2+2} \\
&\quad + 2(n_2-3)\,\varphi_{n_2+2,n_2+2} + 4(n_2-2)\,\varphi_{n_2+2,2n_2+1} + 2(n_2-2)^2\,\varphi_{n_2+2,2n_2+2} \\
&\quad + 4(n_1-3)\,\varphi_{2n_2+1,2n_2+1} + [2(n_1-2)+4(n_1-3)(n_2-2)]\varphi_{2n_2+1,2n_2+2} \\
&\quad + [(n_1-2)(n_2-3)+(n_1-3)(n_2-2)^2]\varphi_{2n_2+2,2n_2+2}.
\end{aligned}$$

This completes the proof. □

Corollary 13. *Let P_{n_1} and P_{n_2} be two path graphs of order n_1 and n_2, respectively. Then*

$$\begin{aligned}
R^t\left(P_{n_1}[P_{n_2}]\right) =& 4[(n_2+1)(n_2+2)]^t + 8[(n_2+1)(2n_2+1)]^t + 4(n_2-2)[(n_2+1)(2n_2+2)]^t \\
&+ 2(n_2-3)(n_2+2)^{2t} + 4(n_2-2)[(n_2+2)(2n_2+1)]^t \\
&+ 2(n_2-2)^2[(n_2+2)(2n_2+2)]^t + 4(n_1-3)(2n_2+1)^{2t} \\
&+ [2(n_1-2)+4(n_1-3)(n_2-2)][(2n_2+1)(2n_2+2)]^t \\
&+ [(n_1-2)(n_2-3)+(n_1-3)(n_2-2)^2](2n_2+2)^{2t}, \\
Z^t\left(P_{n_1}[P_{n_2}]\right) =& 4[(n_2+1)^{t-1}+(n_2+2)^{t-1}] + 8[(n_2+1)^{t-1}+(2n_2+1)^{t-1}] \\
&+ 4(n_2-2)[(n_2+1)^{t-1}+(2n_2+2)^{t-1}] + 4(n_2-3)(n_2+2)^{t-1} \\
&+ 4(n_2-2)[(n_2+2)^{t-1}+(2n_2+1)^{t-1}] + 2(n_2-2)^2[(n_2+2)^{t-1} \\
&+ (2n_2+2)^{t-1}] + 8(n_1-3)(2n_2+1)^{t-1} \\
&+ [2(n_1-2)+4(n_1-3)(n_2-2)][(2n_2+1)^{t-1}+(2n_2+2)^{t-1}] \\
&+ 2(2n_2+2)^{t-1}[(n_1-2)(n_2-3)+(n_1-3)(n_2-2)^2], \\
\chi^t\left(P_{n_1}[P_{n_2}]\right) =& 4(2n_2+3)^t + 8(3n_2+2)^t + 8\cdot 3^t\cdot(n_2-2)(n_2+1)^t + 2(n_2-3)(2n_2+4)^t \\
&+ 2(n_2-2)^2(3n_2+4)^t + 4(n_1-3)(4n_2+2)^t \\
&+ [2(n_1-2)+4(n_1-3)(n_2-2)](4n_2+3)^t \\
&+ 4^t(n_2+1)^t[(n_1-2)(n_2-3)+(n_1-3)(n_2-2)^2]
\end{aligned}$$

for $n_1 \geq n_2 \geq 3$.

Table 12. The basic information on $G_r[P_{n_2}]$.

m_{rn_2+1,rn_2+1}	m_{rn_2+1,rn_2+2}	m_{rn_2+2,rn_2+2}
$2rn_1$	$2n_1[1+r(n_2-2)]$	$n_1(n_2-3)+\frac{rn_1(n_2-2)^2}{2}$

Theorem 14. *Let P_{n_1} and G_r be a path and a r-regular of order n_1 and n_2, respectively. Then*

$$DTI\left(P_{n_1}[G_r]\right) = rn_2\varphi_{r+n_2,r+n_2} + 2n_2^2\varphi_{r+n_2,2n_2+r} + \frac{1}{2}\left[rn_2(n_1-2)+2(n_1-3)n_2^2\right]\varphi_{2n_2+r,2n_2+r}$$

for $n_1 > n_2 \geq 2$.

Proof. By the definition of lexicographic product, we obtain the basic information on $P_{n_1}[G_r]$ in the following Table 11.

Table 11. The basic information on $P_{n_1}[G_r]$.

$m_{r+n_2,r+n_2}$	$m_{r+n_2,2n_2+r}$	$m_{2n_2+r,2n_2+r}$
rn_2	$2n_2^2$	$\frac{rn_2(n_1-2)}{2}+n_2^2(n_1-3)$

Thus we have

$$\begin{aligned} DTI\left(P_{n_1}[G_r]\right) &= \sum_{(i,j)\in K} m_{i,j}(G)\,\varphi_{i,j} \\ &= rn_2\varphi_{r+n_2,r+n_2} + 2n_2^2\varphi_{r+n_2,2n_2+r} + \frac{1}{2}\left[rn_2(n_1-2)+2(n_1-3)n_2^2\right]\varphi_{2n_2+r,2n_2+r}. \end{aligned}$$

This completes the proof. □

Corollary 14. *Let P_{n_1} and G_r be a path and a r-regular of order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(P_{n_1}[G_r]\right) &= rn_2(r+n_2)^{2t} + 2n_2^2(r+n_2)^t(2n_2+r)^t + \left[\frac{rn_2(n_1-2)}{2}+(n_1-3)n_2^2\right](2n_2+r)^{2t}, \\ Z^t\left(P_{n_1}[G_r]\right) &= 2rn_2(r+n_2)^{t-1} + 2n_2^2\left[(r+n_2)^{t-1}+(2n_2+r)^{t-1}\right] + \left[rn_2(n_1-2)+2(n_1-3)n_2^2\right](2n_2+r)^{t-1}, \\ \chi^t\left(P_{n_1}[G_r]\right) &= 2^t rn_2(r+n_2)^t + 2n_2^2(2r+3n_2)^t + 2^t\left[\frac{rn_2(n_1-2)}{2}+(n_1-3)n_2^2\right](2n_2+r)^t \end{aligned}$$

for $n_1 > n_2 \geq 2$.

Theorem 15. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$DTI\left(G_r[P_{n_2}]\right) = 2rn_1\varphi_{rn_2+1,rn_2+1} + 2n_1\left[1+r(n_2-2)\right]\varphi_{rn_2+1,rn_2+2} + \frac{1}{2}\left[2n_1(n_2-3)+rn_1(n_2-2)^2\right]\varphi_{rn_2+2,rn_2+2}$$

for $n_1 \geq n_2 \geq 3$.

Proof. By the definition of lexicographic product, we obtain the basic information on $G_r[P_{n_2}]$ in the following Table 12. Thus we have

$$\begin{aligned} DTI\left(G_r[P_{n_2}]\right) &= \sum_{(i,j)\in K} m_{i,j}(G)\,\varphi_{i,j} \\ &= 2rn_1\varphi_{rn_2+1,rn_2+1} + 2n_1\left[1+r(n_2-2)\right]\varphi_{rn_2+1,rn_2+2} + \frac{1}{2}\left[2n_1(n_2-3)+rn_1(n_2-2)^2\right]\varphi_{rn_2+2,rn_2+2}. \end{aligned}$$

This completes the proof. □

Corollary 15. *Let G_r and P_{n_2} be a r-regular and a path of order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(G_r\left[P_{n_2}\right]\right) =& 2rn_1\left(rn_2+1\right)^{2t}+2n_1\left[1+r\left(n_2-2\right)\right]\left[\left(rn_2+1\right)\left(rn_2+2\right)\right]^t \\ &+\left[n_1\left(n_2-3\right)+\frac{rn_1\left(n_2-2\right)^2}{2}\right]\left(rn_2+2\right)^{2t}, \\ Z^t\left(G_r\left[P_{n_2}\right]\right) =& 4rn_1\left(rn_2+1\right)^{t-1}+2n_1\left[1+r\left(n_2-2\right)\right]\left[\left(rn_2+1\right)^{t-1}+\left(rn_2+2\right)^{t-1}\right] \\ &+\left[2n_1\left(n_2-3\right)+rn_1\left(n_2-2\right)^2\right]\left(rn_2+2\right)^{t-1}, \\ \chi^t\left(G_r\left[P_{n_2}\right]\right) =& 2^{t+1}rn_1\left(rn_2+1\right)^t+2n_1\left[1+r\left(n_2-2\right)\right]\left(2rn_2+3\right)^t \\ &+2^t\left[n_1\left(n_2-3\right)+\frac{rn_1\left(n_2-2\right)^2}{2}\right]\left(rn_2+2\right)^t \end{aligned}$$

for $n_1 \geq n_2 \geq 3$.

Theorem 16. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$DTI\left(G_1[G_2]\right)=\frac{1}{2}n_1n_2\left(r_2+r_1n_2\right)\varphi_{r_1n_2+r_2,r_1n_2+r_2}$$

for $n_1 \geq n_2 \geq 2$.

Proof. By the definition of lexicographic product, we have $G_1[G_2]$ is a $(r_1n_2+r_2)$-regular graph. Thus

$$DTI\left(G_1[G_2]\right)=\sum_{(i,j)\in K} m_{i,j}(G)\,\varphi_{i,j}=\frac{1}{2}n_1n_2\left(r_2+r_1n_2\right)\varphi_{r_1n_2+r_2,r_1n_2+r_2}.$$

This completes the proof. □

Corollary 16. *Let G_1 and G_2 be a r_1-regular graph and a r_2-regular graph with order n_1 and n_2, respectively. Then*

$$\begin{aligned} R^t\left(G_1[G_2]\right) =& \frac{n_1n_2\left(r_2+r_1n_2\right)^{2t+1}}{2}, \\ Z^t\left(G_1[G_2]\right) =& n_1n_2\left(r_2+r_1n_2\right)^t, \\ \chi^t\left(G_1[G_2]\right) =& 2^{t-1}n_1n_2\left(r_2+r_1n_2\right)^{t+1} \end{aligned}$$

for $n_1 \geq n_2 \geq 2$.

6. Conclusion

In this paper, we give a unified approach to solve the computational problems of degree-based topological indices of standard product graphs for the path, star and regular graphs. It is imaginable to use other graph operations to calculate degree-based topological indices uniformly in the future.

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