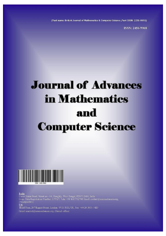

Journal of Advances in Mathematics and Computer Science

35(2): 84-100, 2020; Article no.JAMCS.55916

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

A Novel Modification of Adomian Decomposition Method for Singular BVPs of Emden-Fowler Type

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2020/v35i230251

Editor(s):

(1) Dr. Wei-Shih Du, National Kaohsiung Normal University, Taiwan.

Reviewers:

(1) Rahmatullah Ibrahim Nuruddeen, Federal University Dutse, Nigeria.

(2) Juan Eduardo Napoles Valdes, Universidad Nacional Del Nordeste, Argentina.

Complete Peer review History: http://www.sdiarticle4.com/review-history/55916

Received: 25 January 2020

Accepted: 31 March 2020

Published: 14 April 2020

Original Research Article

Abstract

In this paper, we apply a novel Modified of Adomian Decomposition Method (MADM) for solving Singular Boundary Value Problems (BVPs) of Emden-Fowler type of higher order. The higher-order Emden-Fowler equation is characterized by two types. In addition, we test the presented method by several linear and nonlinear examples, and compared the numerical result with the exact solution to illustrate performance and reliability of this method in finding approximate solutions as well as its successful in getting the complete solution in many case.

Keywords: A Novel Modified of ADM; Higher-order Emden-Fowler Equations; Singular boundary value problems.

1 Introduction

The Emden-Fowler equation is singular differential equations which have great importance in mathematics and other sciences such as fluid mechanics, quantum mechanics, chemical reactor hypothesis and geophysics. Therefore, many scientists have sought to solve this type of equations

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and presented many method as cubic spline method [1], cubic B-spline method [2], Hermite functions collocation method [3], homotopy perturbation method [4], Haar wavelet collocation method [5], the Variational iteration method [6].

In the year 1980s, the Adomian Decomposition Method (ADM) appeared by the American scientist George [7,8,9]. This method solved many equations that the traditional methods were unable to solve.

Many studied that used this method showed the efficiency and effectiveness of this method in finding approximate solution of different types of equations. The ADM yields a very rapid convergence of the solution series in most cases, usually only few iterations leading to very accurate solutions. The ADM has been applied to solve nonlinear singular boundary value problems for ordinary differential equations by many researchers [10,11,12]. In this research, we examine the higher order Emden-Fowler differential equations in the form

$$y^{(n+1)} + \frac{m+n-r}{x} y^{(n)} + \frac{(n-r)(m-1)}{x^2} y^{(n-1)} + f(x,y) = g(x), \tag{1}$$

where $f(x,y)$ and $g(x)$ are known functions, $n \geq 1$ and $r \in \{0,1,...,n\}$. Due to the existence of the singularity at x = 0, such problems show difficulties in finding the solution to this equation. We aim in this work to handle this type of higher order singular BVPs of Emden-Fowler type to find approximate solutions by novel (MADM) which we introduce in this paper. For this reason, we proposed a new differential operator and its inverse operator to solve two different types of Emden-Fowler equations.

2 Higher-Order Emden-Fowler Equation

In the part, we characterize Emden-Fowler of higher order by two types, to derive the Emden-Fowler type equations of higher order Eq.(1), we use the equation

$$x^{-2} \frac{d^{n-r}}{dx^{n-r}} x^{2+n-m-r} \frac{d}{dx} x^{m-n+r} \frac{d^r}{dx^r}(y) + f(x,y) = g(x), \tag{2}$$

where $n - r \geq 0$. To find such distinct order Emden-Fowler equation we put n to distinct values.

First type: for $m \neq 1, \ n \neq r$,

$$y^{(n+1)} + \frac{m+n-r}{x} y^{(n)} + \frac{(n-r)(m-1)}{x^2} y^{(n-1)} + f(x,y) = g(x), \tag{3}$$

under one of the followig

$$y(0) = a_0, y'(0) = a_1, ..., y^{(n-1)}(0) = a_{n-1}, y^{(r)}(b) = a_n, \tag{4}$$

when $n \geq 1, \ 0 \leq r \leq n, \ m \leq (n-r)$.
Or

$$y(b_1) = d_1, y'(b_2) = d_2, ..., y^{(r-1)}(b_{n-1}) = d_{n-1}, y^{(r)}(0) = d_n,$$
$$y^{(r+1)}(0) = d_{n+1}. \tag{5}$$

When $n \geq 1, \ 1 \leq r \leq n-1, \ m \geq 0$.

Where the functions $f(x,y)$, $g(x)$ are known and $a_0, a_1, ..., a_{n-1}, a_n, b, b_0, b_1, ...,$ $b_{n-1}, d_0, d_1, ..., d_{n-1}, d_n$ are constants.

Second type: when we put $n = r$ in (2), we have

$$y^{(n+1)} + \frac{m}{x} y^{(n)} + Ny = g(x). \tag{6}$$

3 Modification of Adomian Method

We rewrite (1) in the form

$$Ly = g(x) - f(x, y), \tag{7}$$

where the new differential operator L is defined by

$$L(.) = x^{-2}\frac{d^{n-r}}{dx^{n-r}}x^{2+n-m-r}\frac{d}{dx}x^{m-n+r}\frac{d^r}{dx^r}(.). \tag{8}$$

Under conditions (4), the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \underbrace{\int_0^x \dots \int_0^x}_{r} x^{n-m-r}\int_b^x x^{m-n-2+r}\underbrace{\int_0^x \dots \int_0^x}_{n-r} x^2(.)dx...dx. \tag{9}$$

The inverse operator under condition (5) is given as

$$L^{-1}(.) = \underbrace{\int_{b_0}^x \dots \int_{b_{n-1}}^x}_{r} x^{n-m-r}\int_0^x x^{m-n-2+r}\underbrace{\int_0^x \dots \int_0^x}_{n-r} x^2(.)dx...dx. \tag{10}$$

Take L^{-1} to both sides of (7) to obtain

$$y(x) = \phi + L^{-1}g(x) - L^{-1}f(x, y). \tag{11}$$

Such that

$$L(\phi) = 0.$$

The ADM assumes that solution y(x) and the nonlinear $f(x, y)$ can be decomposed into an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{12}$$

and

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \tag{13}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and the A_n are the Adomian polynomials, specific algorithms were seen in [13] to formulate Adomian polynomials. The flowing algorithm:

$$\begin{aligned}
A_0 &= F(y_0),\\
A_1 &= F'(y_0)y_1,\\
A_2 &= F'(y_0)y_{(2)} + \frac{1}{2}F''(y_0)y_1^2,\\
A_3 &= F'(y_o)y_{(3)} + F''(y_0)y_1y_2 + \frac{1}{3!}F'''(y_0)y_1^3,\\
&\dots
\end{aligned} \tag{14}$$

Can be used to construct Adomain polynomials, when $F(y)$ is a nonlinear function. By substituting (12) and (13) into (11), we have

$$\sum_{n=0}^{\infty} y_{(n)} = \phi(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{15}$$

Through using ADM, the components y_n can be determined as

$$y_0 = \phi(x) + L^{-1}g(x),$$

$$y_{n+1} = -L^{-1}A_n,\ n \geq 0, \tag{16}$$

which gives

$$\begin{aligned} y_0 &= \phi(x) + L^{-1}g(x), \\ y_1 &= -L^{-1}A_0, \\ y_2 &= -L^{-1}A_1, \\ y_3 &= -L^{-1}A_3, \\ &\ldots \end{aligned} \tag{17}$$

From (14) and (17), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (15) can be immediately obtained. For numerical purposes, the n-term approximate

$$\Psi_n = \sum_{n=0}^{n-1} y_n(x),$$

can be used to approximate the exact solution.

4 Applications

In this section, we will study some example of Emden-Fowler equations with boundary conditions by using the presented technique in this paper.

Example 1. When n=3, r=0, m=2 in (3), we obtain the Emden-Fowler type equation

$$y^{(4)} + \frac{5}{x}y^{(3)} + \frac{3}{x^2} - y = 180 - x^4, \tag{18}$$

$$y(0) = 0, y'(0) = 0, y''(0) = 0, y(1) = 1,$$

where

$$L(.) = x^{-2}\frac{d^3}{dx^3}x^3\frac{d}{dx}x^{-1}(.).$$

So

$$L^{-1}(.) = x\int_1^x x^{-3}\int_0^x\int_0^x\int_0^x x^2(.)dxdxdx.$$

Rewrite Eq. (18) in ADM operator form

$$Ly = 180 - x^4 + y. \tag{19}$$

By using L^{-1} on both sides of (19) we get

$$y = 35.3377x^4 + L^{-1}(180 - x^4) + L^{-1}y.$$

To find the solution, we use the iterative formula

$$\begin{aligned} y_0 &= x + L^{-1}(180 - x^4), \\ y_{n+1} &= L^{-1}y_n,\ n \geq 0, \end{aligned} \tag{20}$$

the first several calculated solution components are

$$y_0 = 0.000283447\,x + x^4 - 0.000283447\,x^8,$$

$$y_1 = -0.000284022\,x + 5.90514\,\times 10^{-7}\,x^5 + 0.000283447\,x^8 - 1.50162\,\times 10^{-8}\,x^{12} + \ldots,$$

$$y_2 = 5.76594\,\times 10^{-7}\,x - 5.91713\,\times 10^{-7}\,x^5 + 1.0252\,\times 10^{-10}\,x^9 + 1.50162\,\times 10^{-8}\,x^{12}$$

$$-2.45364\ \times 10^{-13}\,x^{16},$$
$$y_3 = -1.09876\ \times 10^{-9}\,x + 1.20124\ \times 10^{-9}\,x^5 - 1.02728\ \times 10^{-10}\,x^9 + 3.91177\ \times 10^{-15}\,x^{13}$$
$$+2.45364\ \times 10^{-13}\,x^{16} - +...,$$
$$y_4 = 2.08445\ \times 10^{-12}\,x - 2.28908\ \times 10^{-12}\,x^5 + 2.08548\ \times 10^{-13}\,x^9 - 3.91972\ \times 10^{-15}\,x^{13}$$
$$+4.99358\ \times 10^{-20}\,x^{17} + \ldots,$$

$\vdots$

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 = 3.9968\,10^{-15}\,x + x^4 - 4.33439\,10^{-15}\,x^5 + 3.96658\,10^{-16}\,x^9$$
$$-7.9423\,10^{-18}\,x^{13} + 4.99358\,10^{-20}\,x^{17} + \ldots.$$

It is easily observed that some terms appear in the first components $y_n(x)$ with opposite signs, such as the term $0.000283447\,x^8$ appear in y_0 and y_1 with opposite sings, whenever we continue finding solution, we reach the exact solution

$$y(x) = x^4.$$

Example 2. When n=3, r=1, m= -1 in (3), we study the next equation

$$y^{(4)} + \frac{1}{x}y^{(3)} - \frac{4}{x^2}y(2) = x^8 - y^2, \tag{21}$$

$$y(0) = 1, y'(0) = 0, y''(0) = 0, y'(1) = 4,$$

where

$$L(.) = x^{-2}\frac{d^2}{dx^2}x^5\frac{d}{dx}x^{-3}\frac{d}{dx}(.).$$

So

$$L^{-1}(.) = \int_0^x x^3 \int_1^x x^{-5} \int_0^x \int_0^x x^2(.)dxdxdxdx.$$

The ADM operator form of Eq.(21) is

$$Ly = x^8 - y^2. \tag{22}$$

By using L^{-1} on both sides of (22) we get

$$y = 0.999763\,x^4 + 0.000473485\,x^{16} + L^{-1}x^8 - L^{-1}y^2.$$

To find the solution, we use the recursive relationship

$$y_0 = 0.999763\,x^4 + 0.000473485\,x^{16},$$
$$y_{n+1} = -L^{-1}A_n,\ n \geq 0, \tag{23}$$

where the nonlinear term y^2 has Adomian polynomials A_n as the following

$$A_0 = y_0^2,$$
$$A_1 = 2y_0y_1, \tag{24}$$
$$A_2 = 2y_0y_2 + y_1^2,$$

so, from (23) and (24) we get

$$y_0 = 0.999763\,x^4 + 0.000473485\,x^{16},$$
$$y_1 = -0.000236652\,x^4 + 0.0000788768\,x^{12} + 3.57316 \times\ 10^{-9}\,x^{24} + 1.5445\,10^{-13}\,x^{36},$$
$$y_2 = 1.05544 \times\ 10^{-7}\,x^4 - 3.73415 \times\ 10^{-8}\,x^{12} + 1.29701 \times\ 10^{-9}\,x^{20} - 8.45796 \times\ 10^{-13}\,x^{24} + \ldots,$$

$$y_3 = -7.05475 \times 10^{-11}\, x^4 + 2.54929 \times 10^{-11}\, x^{12} - 1.22805 \times 10^{-12}\, x^{20} + 3.77215 \times 10^{-16}\, x^{24} + \ldots,$$

$$\vdots$$

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 = x^4 - 0.0000788395\, x^{12} + 0.000473485\, x^{16} - 1.29578 \times 10^{-9}\, x^{20} - 3.57232 \times 10^{-9}\, x^{24} - \ldots.$$

In Table 1, we give the exact solutions and the ADM solution in [0,1].

Table 1. Numerical results for Example2

x	Exact solution	MADM solution	Absolute Error
0.0	0.000000000	0.00000000	000000000
0.1	0.0001	0.0001	1.96×10^-11
0.2	0.0016	0.008099	3.14×10^-10
0.3	0.0081	0.03643534	1.62×10^-9
0.4	0.0256	0.025599	6.13×10^-9
0.5	0.0625	0.062499	2.42×10^-8
0.6	0.24009	0.2401004	6.34×10^-8
0.7	0.69071717	0.69038295	4.35×10^-7
0.8	0.4096	0.409608	7.82925×10^-6
0.9	0.6561	0.656165	0.00006534
1	1	1.000394	0.0003944

Example 3. When n=3, r=1, m=5 in (3), we obtain the Emden-Fowler type equation

$$y^{(4)} + \frac{7}{x} y^{(3)} + \frac{8}{x^2} y^{(2)} = 288 + x^8 - y^2, \tag{25}$$

$$y(\frac{1}{2}) = 0.0625, y'(0) = 0, y''(0) = 0, y^3(0) = 0,$$

where

$$L(.) = x^{-2} \frac{d^2}{dx^2} x^{-1} \frac{d}{dx} x^3 \frac{d}{dx}(.).$$

So

$$L^{-1}(.) = \int_{\frac{1}{2}}^{x} x^{-3} \int_0^x x \int_0^x \int_0^x x^2 (.) dx dx dx dx.$$

Rewrite Eq.(25) in ADM operator form

$$Ly = 288 + x^8 - y^2. \tag{26}$$

By using L^{-1} on both sides of (26) we get

$$y = -1.10092\, 10^{-8} + x^4 + 0.0000450938\, x^{12} - L^{-1} y^2.$$

To find the solution, we use the iterative formula

$$y_0 = -1.10092\, 10^{-8} + x^4 + 0.0000450938\, x^{12},$$

$$y_{n+1} = -L^{-1} A_n,\ n \geq 0, \tag{27}$$

where the nonlinear term y^2 has Adomian polynomials A_n as the following

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1, \qquad (28)$$
$$A_2 = 2y_0y_2 + y_1^2,$$

so, from (27) and (28) we get

$$y_0 = -1.10092 \times 10^{-8} + x^4 + 0.0000450938\, x^{12},$$

$$y_1 = -1.10092 \times 10^{-8} + 4.20844 \times 10^{-19}\, x^4 - 4.91483 \times 10^{-12}\, x^8 + 0.0000450938\, x^{12} + \ldots,$$
$$y_2 = 1.86841 \times 10^{-14} + 8.41687 \times 10^{-19}\, x^4 - 4.91483 \times 10^{-12}\, x^8 + 4.28348 \times 10^{-23}\, x^{12} - \ldots,$$
$$y_3 = -5.65141 \times 10^{-20} + 4.20841 \times 10^{-19}\, x^4 + 8.3411 \times 10^{-18}\, x^8 + 8.56695 \times 10^{-23}\, x^{12} - \ldots,$$
$$\vdots$$

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 = -3.73681 \times 10^{-14} + x^4 + 9.82965 \times 10^{-12}\, x^8 + 3.41882 \times 10^{-16}\, x^{16} + \ldots.$$

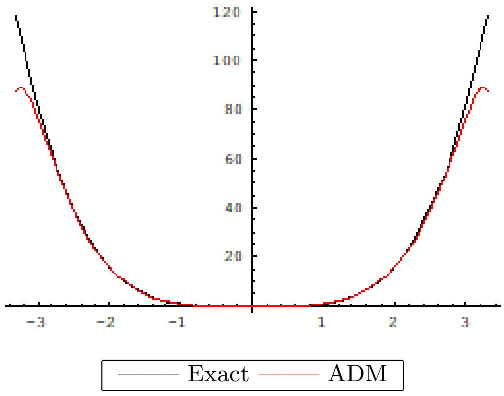

Fig. 1. The exact solution $y = x^4$, and the MADM solution $y = \sum_{n=0}^{3} y_n(x)$.

Example 4. When n=3, r=2, m=-3 in (3), we obtain the Emden-Fowler type equation

$$y^{(4)} - \frac{2}{x}y^{(3)} - \frac{4}{x^2}y^{(2)} = -8\left(9 + 85\, x^4 - 113\, x^8 + 3\, x^{12}\right) e^{-4y}, \qquad (29)$$

$$y(0) = 0, y'(0) = 0, y''(0) = 0, y''(1) = 2,$$

the exact solution is $y(x) = \log(1 + x^4)$, where

$$L(.) = x^{-2}\frac{d}{dx}x^6\frac{d}{dx}x^{-4}\frac{d^2}{dx^2}(.).$$

So

$$L^{-1}(.) = \int_0^x \int_0^x x^4 \int_1^x x^{-6} \int_0^x x^2(.)dxdxdxdx.$$

The ADM operator form of Eq.(29) is

$$Ly = -8\left(9 + 85\, x^4 - 113\, x^8 + 3\, x^{12}\right) e^{-4y}. \qquad (30)$$

By using L^{-1} on both sides of (30) we get

$$y = 0.0666667x^6 - L^{-1}(8\left(9 + 85\, x^4 - 113\, x^8 + 3\, x^{12}\right) e^{-4}).$$

To find the solution, we use the iterative formula

$$\begin{aligned} y_0 &= 0.0666667x^6, \\ y_{n+1} &= -L^{-1}(8\left(9+85\,x^4-113\,x^8+3\,x^{12}\right))A_n,\ n\geq 0, \end{aligned} \tag{31}$$

where the nonlinear term e^{-4y} has Adomian polynomials A_n as the following

$$\begin{aligned} A_0 &= e^{-4y_0}, \\ A_1 &= -4y_1e^{-4y_0}, \\ A_2 &= 4(-y_2+2y_1^2)e^{-4y_0}, \end{aligned} \tag{32}$$

so, from (31) and (32) we get

$$\begin{aligned} y_0 &= 0.0666667x^6, \\ y_1 &= x^4+0.731732\,x^6-0.867347\,x^8-0.00592593\,x^{10}+0.103765\,x^{12}-\ldots, \\ y_2 &= -1.36951\,x^6+0.367347\,x^8+0.0650428\,x^{10}+0.28354\,x^{12}+\ldots, \\ y_3 &= 0.648206\,x^6-0.121735\,x^{10}-0.053972\,x^{12}-\ldots, \end{aligned}$$

$\vdots$

The solution in a series form is given by

$$y(x) = y_0+y_1+y_2+y_3 = x^4+0.0770907\,x^6-0.5\,x^8-0.0626176\,x^{10}+0.333333\,x^{12}-\ldots.$$

Where the first terms of the exact solution series are

$$y(x) = x^4-0.5\,x^8+0.333333\,x^{12}-0.25\,x^{16}+\ldots.$$

In Table 2, we give the exact solutions and the MADM solution in [0,1].

Table 2. Numerical results for Example 4

x	True solution	ADM solution	Absolute Error
0.0	0.000000000	0.00000000	000000000
0.1	0.000099	0.0001	$7.7\times10^{-}8$
0.2	0.001598	0.001603	$4.92\times10^{-}6$
0.3	0.008067	0.008123	0.0000558
0.4	0.025277	0.025586	0.000309
0.5	0.060624	0.061764	0.00114
0.6	0.121864	0.125043	0.003179
0.7	0.215192	0.222225	0.007032
0.8	0.343306	0.355584	0.012278
0.9	0.504465	0.520206	0.0157402
1	0.693147	0.70287	0.009723

Example 5. When n=3, r=2, m=3 in (3), we study the Emden-Fowler type equation

$$y^{(4)}+\frac{4}{x}y^{(3)}+\frac{2}{x^2}y^{(2)} = 16\,\left(9-62\,x^4+25\,x^8\right)e^{-4y}, \tag{33}$$

$$y(0)=0, y'(0.1)=0.0039996, y''(0)=0, y^3(0)=0,$$

with exact solution $\log(1+x^4)$ where

$$L(.) = x^{-2}\frac{d^2}{dx^2}x^2\frac{d^2}{dx^2}(.).$$

So

$$L^{-1}(.) = \int_0^x \int_{0.1}^x x^{-2} \int_0^x \int_0^x x^2 (.) dxdxdxdx.$$

Rewrite Eq.(33) in ADM operator

$$Ly = 16 \left(9 - 62\,x^4 + 25\,x^8\right) e^{-4y}. \tag{34}$$

By using L^{-1} on both sides of (34) we get

$$y = 0.0666667x^6 + L^{-1}x^8 - L^{-1}(16 \left(9 - 62\,x^4 + 25\,x^8\right))e^{-4}.$$

To find the solution, we use the iterative formula

$$y_0 = 0.0039996\,x,$$
$$y_{n+1} = -L^{-1}(16 \left(9 - 62\,x^4 + 25\,x^8\right))A_n,\ n \geq 0, \tag{35}$$

where the nonlinear term e^{-4y} has Adomian polynomials A_n as the following

$$A_0 = e^{-4y_0},$$
$$A_1 = -4y_1 e^{-4y_0}, \tag{36}$$
$$A_2 = 4(-y_2 + 2y_1^2)e^{-4y_0},$$

so, from (35) and (36) we get

$$y_0 = 0.0039996\,x,$$
$$y_1 = -0.00399687\,x + x^4 - 0.00575942\,x^5 + 0.0000204759\,x^6 - 5.57112\,10^{-8}\,x^7$$
$$-0.316327\,x^8 + 0.00306142\,x^9 - 0.0000156729\,x^{10} + 5.59506\,10^{-8}\,x^{11} + 0.0229568\,x^{12} - \ldots,$$
$$y_2 = -2.72832\,10^{-6}\,x + 0.00575549\,x^5 - 0.0000409238\,x^6 + 1.67019\,10^{-7}\,x^7 - 0.183673\,x^8$$
$$-0.000641796\,x^9 + 0.0000142157\,x^{10} - 8.19173\,10^{-8}\,x^{11} + 0.238189\,x^{12} - \ldots,$$
$$y_3 = -2.97968\,10^{-9}\,x + 3.92879\,10^{-6}\,x^5 + 0.00002042\,x^6 - 1.66791\,10^{-7}\,x^7 + 7.50657\,10^{-10}\,x^8$$
$$-0.00241797\,x^9 + 0.000018564\,x^{10} - 8.97757\,10^{-8}\,x^{11} + 0.0721876\,x^{12} + \ldots,$$

$\vdots$

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 = 3.91016\,10^{-12}\,x + x^4 - 4.29638\,10^{-9}\,x^5 - 2.79257\,10^{-8}\,x^6 - 5.54831\,10^{-8}\,x^7$$
$$-0.5\,x^8 + 1.65321\,10^{-6}\,x^9 + 0.0000171068\,x^{10} - 1.15742\,10^{-7}\,x^{11} + 0.333333\,x^{12} - \ldots.$$

Where the first terms of the exact solution series are

$$y(x) = x^4 - 0.5\,x^8 + 0.333333\,x^{12} - 0.25\,x^{16} + \ldots.$$

In Table 3, we give the exact solutions and the ADM solution in [0,1].

Table 3. Numerical results for Example 5

x	Exact solution	ADM solution	Absolute Error
0.0	0.000000000	0.00000000	000000000
0.1	0.000099	0.000099	3.17×10^{-13}
0.2	0.001598	0.001598	1.43×10^{-12}
0.3	0.008067	0.008067	1.22×10^{-11}
0.4	0.025277	0.025277	6.47×10^{-9}
0.5	0.060624	0.060624	2.85×10^{-7}
0.6	0.0.121864	0.121868	4.54×10^{-6}
0.7	0.215192	0.215223	0.00003069
0.8	0.343306	0.343312	6.48×10^{-6}
0.9	0.504465	0.0.502894	0.0015709
1	0.693147	0.678501	0.014646

Example 6. When n=4, r=3, m=-2 in (3), we obtain the Emden-Fowler type equation

$$y^{(5)} - \frac{1}{x}y^{(4)} - \frac{3}{x^2}y(3) = \frac{-48\left(64+80\,x^2-44\,x^4+x^6\right)}{x\,(4+x^2)}e^{4y}, \tag{37}$$

$$y(0) = -1.38629, y'(0) = 0, y''(0) = -\frac{1}{2}, y'''(0) = 0, y'''(2) = \frac{1}{8},$$

with exact solution $\log(\frac{1}{4+x^2})$ where

$$L(.) = x^{-2}\frac{d}{dx}x^5\frac{d}{dx}x^{-3}\frac{d^3}{dx^3}(.).$$

So

$$L^{-1}(.) = \int_0^x\int_0^x\int_0^x x^3\int_2^x x^{-5}\int_0^x x^2(.)dxdxdxdx.$$

In an operator form, Eq.(37) becomes

$$Ly = \frac{-48\left(64+80\,x^2-44\,x^4+x^6\right)}{x\,(4+x^2)}e^{4y}. \tag{38}$$

By using L^{-1} on both sides of (38) we get

$$y = -1.38629 - 0.25\,x^2 + 0.000181159\,x^6 - L^{-1}\frac{48\left(64+80\,x^2-44\,x^4+x^6\right)}{x\,(4+x^2)}e^4.$$

To find the solution, we use the iterative formula

$$\begin{aligned} y_0 &= -1.38629 - 0.25\,x^2 + 0.000181159\,x^6, \\ y_{n+1} &= -L^{-1}\frac{48\left(64+80\,x^2-44\,x^4+x^6\right)}{x\,(4+x^2)}A_n,\ n \geq 0, \end{aligned} \tag{39}$$

where the nonlinear term e^{4y} has Adomian polynomials A_n as the following

$$\begin{aligned} A_0 &= e^{4y_0}, \\ A_1 &= 4y_1e^{4y_0}, \\ A_2 &= 4(y_2e^{4y_0} + 2e^{4y_0}y_1^2), \\ &\dots \end{aligned} \tag{40}$$

So, from (39) and (40) we get

$$\begin{aligned} y_0 &= -1.38629 - 0.25\,x^2 + 0.000181159\,x^6, \\ y_1 &= 0.03125\,x^4 - 0.00591649\,x^6 + 0.00106957\,x^8 - 0.00019812\,x^{10} + 0.0000343277\,x^{12} - \dots, \\ y_2 &= 0.000151839\,x^6 - 0.000093006\,x^8 + 3.0815\,10^{-6}\,x^{10} + 6.64429\,10^{-6}\,x^{12} - \dots, \\ y_3 &= -0.0000707257\,x^6 - 7.90828\,10^{-8}\,x^{10} - 2.81836\,10^{-7}\,x^{12} + \dots, \end{aligned}$$

$\vdots$

The solution in a series form is given by

$$\begin{aligned} y(x) = y_0 + y_1 + y_2 + y_3 = {} & -1.38629 - 0.25\,x^2 + 0.03125\,x^4 - 0.00565421\,x^6 + 0.000976563\,x^8 \\ & -0.000195117\,x^{10} + 0.0000406901\,x^{12} - \dots. \end{aligned}$$

Where the first terms of the exact solution series are

$$y(x) = -1.38629 - 0.25\,x^2 + 0.03125\,x^4 - 0.00520833\,x^6 + 0.000976563\,x^8 - 0.000195313\,x^{10}$$

$$+0.0000406901\,x^{12} - \ldots.$$

Example 7. When n=4, r=3, m=2 in (3), we obtain the Emden-Fowler type equation

$$y^{(5)} + \frac{3}{x}y^{(4)} + \frac{1}{x^2}y^{(3)} = \frac{-16\,\left(-192+400\,x^2-76\,x^4+x^6\right)}{x\,(4+x^2)^5}e^{4y}, \tag{41}$$

$$y(0) = -1.38629, y'(\frac{1}{2}) = -\frac{4}{17}, y''(1) = -\frac{6}{25}, y'''(0) = 0, y''''(0) = 0,$$

with exact solution $\log(\frac{1}{4+x^2})$ where

$$L(.) = x^{-2}\frac{d}{dx}x\frac{d}{dx}x\frac{d^3}{dx^3}(.).$$

So

$$L^{-1}(.) = \int_0^x\int_{\frac{1}{2}}^x\int_1^x x^{-1}\int_0^x x^{-1}\int_0^x x^2(.)dxdxdxdx.$$

Rewrite Eq.(41) in ADM operator form

$$Ly = \frac{-16\,\left(-192+400\,x^2-76\,x^4+x^6\right)}{x\,(4+x^2)^5}e^{4y}. \tag{42}$$

By using L^{-1} on both sides of (42) we get

$$y = -1.38629 - 0.115294\,x - 0.12\,x^2 - L^{-1}\frac{16\,\left(-192+400\,x^2-76\,x^4+x^6\right)}{x\,(4+x^2)^5}e^4.$$

To find the solution, we use the iterative formula

$$y_0 = -1.38629 - 0.115294\,x - 0.12\,x^2,$$

$$y_{n+1} = -L^{-1}\frac{16\,\left(-192+400\,x^2-76\,x^4+x^6\right)}{x\,(4+x^2)^5}A_n,\ n \geq 0, \tag{43}$$

where the nonlinear term e^{4y} has Adomian polynomials A_n as the following

$$\begin{aligned} A_0 &= e^{4y_0}, \\ A_1 &= 4y_1e^{4y_0}, \\ A_2 &= 4(y_2e^{4y_0} + 2e^{4y_0}y_1^2), \\ &\ldots \end{aligned} \tag{44}$$

So, from (43) and (44) we get

$$y_0 = -1.38629 - 0.115294\,x - 0.12\,x^2,$$

$$y_1 = 0.107496\,x - 0.121633\,x^2 + 0.03125\,x^4 - 0.00256209\,x^5 - 0.00422967\,x^6 + 0.000732055\,x^7$$
$$+0.000475465\,x^8 - \ldots,$$

$$y_2 = 0.00696894\,x - 0.00748561\,x^2 + 0.00238881\,x^5 - 0.00107005\,x^6 - 0.000536909\,x^7$$
$$+0.000494267\,x^8 + \ldots,$$

$$y_3 = 0.000723344\,x - 0.000770121\,x^2 + 0.000154865\,x^5 + 0.0000775716\,x^6 - 0.000179134\,x^7$$
$$+0.0000201773\,x^8 + \ldots,$$

$\vdots$

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 = -1.38629 - 0.000105401\,x - 0.249888\,x^2 + 0.03125\,x^4 - 0.0000184166\,x^5$$
$$-0.00522215\,x^6 + 0.0000160112\,x^7 + 0.00098991\,x^8 - \dots.$$

Example 8. When n=5, r=4, m=7 in (3), we obtain the Emden-Fowler type equation

$$y^{(6)} + \frac{8}{x}y^{(5)} + \frac{6}{x^2}y^{(4)} = y^2 - x^6, \tag{45}$$

$$y(0) = 0, y'(0.5) = 0.75, y''(0.1) = 0.6, y'''(0.2) = 6, y^{(4)}(0) = 0, y^{(5)}(0) = 0,$$

the exact solution is $y(x) = x^3$
re-written Eq.(45), as

$$Ly = y^2 - x^6, \tag{46}$$

we give

$$L(.) = x^{-2}\frac{d}{dx}x^{-4}\frac{d}{dx}x^6\frac{d^4}{dx^4}(.).$$

The inverse operator

$$L^{-1}(.) = \int_0^x\int_{0.5}^x\int_{0.1}^x\int_{0.2}^x x^{-6}\int_0^x x^4\int_0^x x^2(.)dxdxdxdxdxdx. \tag{47}$$

Applying L^{-1} on both sides of (46), we get

$$y(x) = 3.88053\,10^{-9}\,x - 2.25705\,10^{-11}\,x^2 + 1.\,x^3 - 6.68056\,10^{-7}\,x^{12} + L^{-1}y_2,$$

using ADM for $y^2(x)$, as yield

$$\sum_{n=0}^{\infty} y_n(x) = 3.88053\,10^{-9}\,x - 2.25705\,10^{-11}\,x^2 + 1.\,x^3 - 6.68056\,10^{-7}\,x^{12} + L^{-1}A_n,\ n \geq 0,$$

the nonlinear term y^2, we get it as

$$A_0 = y_0^2,$$
$$A_1 = 2y_0y_1,$$

the first few components are as follows

$$y_0 = 3.88053\,\times 10^{-9}\,x - 2.25705\,\times 10^{-11}\,x^2 + \,x^3 - 6.68056\,\times 10^{-7}\,x^{12},$$

$$y_1 = -3.88053\,\times 10^{-9}\,x + 2.25706\,\times 10^{-11}\,x^2 - 7.52499\,\times 10^{-11}\,x^3 + 1.79268\,\times 10^{-22}\,x^8$$
$$-8.77681\,\times 10^{-25}\,x^9 + 1.8332\,\times 10^{-14}\,x^{10} - 5.48042\,\times 10^{-17}\,x^{11} + 6.68056\,\times 10^{-7}\,x^{12} - \dots,$$
$$y_2 = 3.44924\,\times 10^{-16}\,x - 8.43561\,\times 10^{-18}\,x^2 + 2.81462\,\times 10^{-17}\,x^3 - 3.58535\,\times 10^{-22}\,x^8$$
$$+1.75536\,\times 10^{-24}\,x^9 - 1.8332\,\times 10^{-14}\,x^{10} + 5.48042\,\times 10^{-17}\,x^{11} - 1.00542\,\times 10^{-16}\,x^{12} + \dots,$$

the solution in a series form are given by

$$y(x) = y_0 + y_1 + y_2 = 4.05319\,\times 10^{-23}\,x - 1.70929\,\times 10^{-24}\,x^2 + \,x^3 - 1.79268\,\times 10^{-22}\,x^8$$
$$+8.77682\,\times 10^{-25}\,x^9 - 1.63084\,\times 10^{-21}\,x^{10} + 2.04868\,10^{-23}\,x^{11} - 1.05879\,\times 10^{-22}\,x^{12} + \dots.$$

Table 4. Comparison of numerical errors between the exact solution and the MADM solutions

x	Exact	MADN	Error
0.1	0.001	0.001	4.03×10^{-24}
0.2	0.008	0.008	8.03×10^{-24}
0.3	0.027	0.027	1.19×10^{-23}
0.4	0.064	0.064	1.56×10^{-23}
0.5	0.125	0.125	1.7×10^{-23}
0.6	0.216	0.216	$1.07\times10-23$
0.7	0.343	0.343	2.83×10^{-23}
0.8	0.512	0.512	1.61×10^{-22}
0.9	0.729	0.729	4.89×10^{-22}
1.0	1.00	1.0000	1.37×10^{-21}

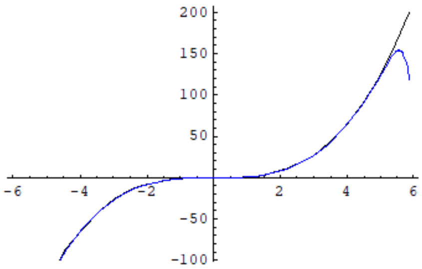

Fig. 2. The exact solution $y = x^3$, and the MADM solution $y = \sum_{n=0}^{2} y_n(x)$.

5 The Second Type of Emden-Fowler Equation of Higher Order

The second Emden-Fowler type equation of n+1 order ordinary differential equation is defined in the form

$$y^{(n+1)} + \frac{m}{x}y^{(n)} + f(x,y) = g(x), \tag{48}$$

under one of the boundary conditions

$$y(0) = d_0, y'(0) = d_1, ..., y^{(n-1)}(0) = d_{n-1}, y^{(n)}(b_n) = d_n. \tag{49}$$

or

$$y(b_0) = a_0, y'(b_1) = a_1, y''(b_2) = a_2, ..., y^{(n-1)}(b_{(n-1)}) = a_{(n-1)}, y^{(n)}(0) = a_n. \tag{50}$$

Where N is the nonlinear operator ,$g(x)$ is real function and $a_0, a_1, ..., a_{n-1}, a_n, b$ are constants.

In an operator form, Eq. (48) can be written as

$$Ly = g(x) - f(x,y), \tag{51}$$

where the differential operator$L$ is

$$L(.) = x^{-m}\frac{d}{dx}x^m\frac{d^n}{dx^n}. \tag{52}$$

When $m \leq 0,\ n \geq 1$, we define the inverse operator L^{-1} by

$$L^{-1}(.) = \underbrace{\int_0^x \dots \int_0^x}_{n} x^{-m} \int_b^x x^m (.) dx \dots dx. \tag{53}$$

When $m \geq 0,\ n \geq 1$, we present the inverse operator

$$L^{-1}(.) = \underbrace{\int_{b_1}^x \dots \int_{b_{n-1}}^x}_{n} x^{-m} \int_0^x x^m (.) dx \dots dx. \tag{54}$$

Take L^{-1} to both sides of (51) to obtain

$$y(x) = \phi + L^{-1} g(x) - L^{-1} f(x, y). \tag{55}$$

Example 9. We consider the Emden-Fowler type equation

$$y^{(5)} + \frac{36}{x} y^{(4)} = \frac{e^x \left(864 + 1356\, x + 492\, x^2 + 51\, x^3 + x^4\right)}{x}, \tag{56}$$

$$y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(3) = 3374.37, y'''(0) = 6,$$

with exact solution $x^3 e^x$ where

$$L(.) = x^{-36} \frac{d}{dx} x^{36} \frac{d^4}{dx^4}(.).$$

So

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_3^x x^{-36} \int_0^x x^{36}(.) dx dx dx dx.$$

Rewrite Eq.(56) in ADM operator form

$$Ly = \frac{e^x \left(864 + 1356\, x + 492\, x^2 + 51\, x^3 + x^4\right)}{x}. \tag{57}$$

By using L^{-1} on both sides of (57) we get the exact solution

$$y = x^3 e^x.$$

Example 10. For n=4, m=3 in (48), consider the Emden-Fowler type equation

$$y^{(5)} + \frac{3}{x} y^{(4)} = 9\, e^{x^3}\, x(100 + 348\, x^3 + 207\, x^6 + 27\, x^9) + e^{2x^3} - y^2, \tag{58}$$

$$y(0.1) = 1.001, y'(0) = 0, y''(0.5) = 4.03684, y'''(0.6) = 23.4863, y''''(0) = 0,$$

with exact solution $y(x) = e^{x^3}$.

We put

$$L(.) = x^{-3} \frac{d}{dx} x^3 \frac{d^4}{dx^4}(.),$$

$$L^{-1}(.) = \int_{0.1}^x \int_0^x \int_{0.5}^x \int_{0.6}^x x^{-3} \int_0^x x^3(.) dx dx dx dx dx.$$

The ADM operator form of Eq.(58) is

$$Ly = 9\, e^{x^3}\, x(100 + 348\, x^3 + 207\, x^6 + 27\, x^9) + e^{2x^3} - y^2, \tag{59}$$

Take L^{-1} on both sides of (59) and using the boundary condition gives

$$y = 1.03562 - 3.85315\,x^2 + 3.91438\,x^3 + L^{-1}(9\,e^{x^3}\,x(100 + 348\,x^3 + 207\,x^6 + 27\,x^9) + e^{2x^3}) - L^{-1}y^2.$$

Proceeding as before we obtained the recursive relationship

$$\begin{gathered} y_0 = 0.999911 + 0.00976473\,x^2 + 0.991703\,x^3 + 0.00208333\,x^5 + 0.5\,x^6 \\ +0.000170068\,x^8 + 0.166667\,x^9 + 0.0000252525\,x^{11} + 0.0416667\,x^{12} + 4.26924\,10^{-6}\,x^{14} \\ +0.00833333\,x^{15} + 0.00135885\,x^{18} + 0.000150376\,x^{21} + 6.90472\,10^{-6}\,x^{24}. \\ y_{n+1} = -L^{-1}A_n,\ n \geq 0, \end{gathered} \tag{60}$$

where A_n are Adomian polynomials of nonlinear term e^{4y}, as below,

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_0y_1, \\ A_2 &= y_1^2 + 2y_0y_2, \end{aligned} \tag{61}$$

$$\dots$$

From (60) and (61) we have

$$\begin{gathered} y_1 = 0.0000897823 - 0.00980894\,x^2 + 0.0083279\,x^3 - 0.00208296\,x^5 - 3.87455\,\times 10^{-6}\,x^7 \\ -0.000168642\,x^8 - 3.94138\times 10^{-9}\,x^9 - 5.18821\times 10^{-7}\,x^{10} - 0.0000250427\,x^{11} - 3.11344\times 10^{-10}\,x^{12} \\ -\dots, \\ y_2 = -1.43588\times 10^{-7} + 0.0000154943\,x^2 - 0.0000113515\,x^3 - 3.7406\times 10^{-7}\,x^5 + 3.89174\times 10^{-6}\,x^7 \\ -1.43132\times 10^{-6}\,x^8 + 7.91846\times 10^{-9}\,x^9 + 5.17144\times 10^{-7}\,x^{10} - 2.09689\times 10^{-7}\,x^{11} \\ +6.83334\times 10^{-10}\,x^{12} + \dots, \\ y_3 = 3.09374\,10^{-10} - 3.33596\,10^{-8}\,x^2 + 2.42159\times 10^{-8}\,x^3 + 5.81438\times 10^{-10}\,x^5 - 5.79796\times 10^{-9}\,x^7 \\ +1.82741\times 10^{-9}\,x^8 - 3.98966\times 10^{-9}\,x^9 + 2.95389\times 10^{-9}\,x^{10} - 5.89592\times 10^{-10}\,x^{11} \\ -3.72686\times 10^{-10}\,x^{12} + \dots, \end{gathered}$$

$$\vdots$$

This means that the solution in a series form is given by

$$\begin{gathered} y = y_0 + y_1 + y_2 + y_3 = 0.99991 - 0.0000287467\,x^2 + 1.00002\,x^3 - 1.11644\,10^{-9}\,x^5 + 0.5\,x^6 \\ +1.13953\,10^{-8}\,x^7 - 3.31116\,10^{-9}\,x^8 + 0.166667\,x^9 + 1.27672\,10^{-9}\,x^{10} - 4.90608\,10^{-10}\,x^{11} \\ +0.0416667\,x^{12} + \dots \end{gathered}$$

In Fig. 3, we have plotted $\sum_0^3 y_i(x)$, which is similar to the true solution $y(x) = e^{x^3}$.

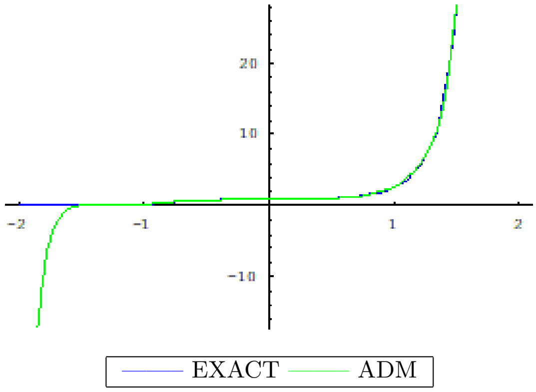

Fig. 3. The exact solution $y = e^{x^3}$ and the MADM solution $y = \sum_{n=0}^{3} y_n(x)$.

6 Conclusion

In this paper, two Emden-Fowler equations are studied, with great applicability in different fields of science and technique. These higher-order equations with boundary conditions are introduced to modified (ADM) to solve. This method is dependable to overcome the difficulty of the singular point at x = 0. Illustrative example were studied to corroborate the efficiency and reliability of the proposed method and to show the rapid convergence of the approximation series as the solution.

Competing Interests

Authors have declared that no competing interests exist.

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