Asian Research Journal of Mathematics

15(4): 1-17, 2019; Article no.ARJOM.52223

ISSN: 2456-477X

On some Ternary Linear Codes and Designs obtained from the Projective Special Linear Goup $PSL_3(4)$

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Authors' contributions

This work was carried out in collaboration among all the authors. YK designed the study, performed the mathematical analysis, wrote the protocol and the first draft of the manuscript. OMO and OMI in collaboration managed the analyses of the study and sharpened some of the proofs of the main results. All the authors read and approved of final manuscript.

Article Information

DOI: 10.9734/ARJOM/2019/v15i430152

Editor(s):

(1) Dr. Hari Mohan Srivastava, Professor, Department of Mathematics and Statistics, University of Victoria, Canada.

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Complete Peer review History: http://www.sdiarticle4.com/review-history/52223

Received: 16 August 2019

Accepted: 24 October 2019

Published: 05 November 2019

Original Research Article

Abstract

Let $(\mathbf{G}, *)$ be a group and $\mathbf{X}$ any set, an action of a group G on X, denoted as G×X → X, $(g, x) \mapsto g.x$, assigns to each element in G a transformation of X that is compatible with the group structure of G. If G has a subgroup H then there is a transitive group action of G on the set (G/H) of the right co-sets of H by right multiplication. A representation of a group G on a vector space V carries the dimension of the vector space. Now, given a field F and a finite group G, there is a bijective correspondence between the representations of G on the finite-dimensional $\mathbb{F}$-vector spaces and finitely generated $\mathbb{F}$G-modules. We use the $\mathbb{F}$G -modules to construct linear ternary codes and combinatorial designs from the permutation representations of the group $L_3(4)$. We investigate the properties and parameters of these codes and designs.

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We further obtain the lattice structures of the sub-modules and compare these ternary codes with the binary codes constructed from the same group.

Keywords: Ternary linear codes; simple groups; designs; linear groups.

2010 Mathematics Subject Classification: Primary 20K30; Secondary 16P10.

1 Introduction

In coding theory, a linear code is an error-correcting code for which any linear combination of code-words is also a codeword. Error correcting codes have become an integral parts in the design of reliable data transmissions and storage systems often due to the noise present that distorts the transmitted information. Coding theory was pioneered by Claude Shannon after his seminal paper; "A mathematical theory of communication" in 1948 (see [1]) and R.W. Hamming who wrote the paper, "Error detecting and error correcting codes" in 1950 (see [2]) where he provided the first examples of error detecting and correcting codes. They introduced a relationship between coding theory and information theory, two previously independent fields of mathematics.

Linear codes can be constructed from finite groups through the modular theoretic approach. The associated permutation modules of the group over the field F_q are determined and thus the subsequent maximal sub-modules. Each of the sub-modules is a q-ary code invariant under the group. This method has been used in the enumeration of binary linear codes from several groups such as $PSU_4(2)$, $PSU_3(3)$ and L_3 (4) as seen in [3]. Further, linear codes can be generated from incidence matrices of the combinatorial designs obtained from the primitive permutation representations of finite simple groups. The combinatorial designs are usually generated from the orbits of the stabilizers of the elements from the set that the group acts on. Some of the codes generated using this approach are from the groups J_1, J_2 and Co_2 (see [4, 5]).

The construction and classification of binary linear codes from simple groups seems to have received more attention compared to the ternary linear codes as seen in [6, 7] among others. Our primary reference works are the study of linear binary codes generated from the simple group L_3 (4) (see[8, 9]). The projective special linear group L_3 (4) is a classical simple group of order 20160 with nine maximal sub groups of degrees 21, 21, 56, 56, 56, 120,120,120 and 280. In an attempt to bridge the gap between the binary and ternary linear codes, we study the ternary linear codes that are obtained from the 3-modular representation of the Group L_3 (4)and compare our results with those of the linear binary codes.

We construct the ternary linear codes from the primitive representation of the group L_3 (4) under the field $G\mathbb{F}(3)$. This method enables us to exhaustively enumerate all the ternary linear codes from the maximal sub-modules of the maximal subgroups of L_3 (4) as well as classify them. We further determine the relationship between these ternary linear codes and the combinatorial designs obtained from the support of the code words of the ternary linear codes obtained from the group. Due to their sizes, we work on the maximal subgroups of the degrees 21, 56 and 120. We also present the obtained lattice diagrams which shall provide an insight to the internal properties of L_3 (4).

2 Fundamental Principles

A simple group G is a group with no proper normal subgroups. In 1982, the finite simple groups were classified into four categories; cyclic groups of prime power order, alternating groups of degree at least five, the Chevalley and twisted Chevalley groups and the 26 sporadic groups [10]. The Chevalley and twisted Chevalley groups, also referred in some cases as the classical groups and groups of Lie type are further classified into; linear, unitary, orthogonal and symplectic groups. The linear group, also referred to as the General Linear Groups, denoted *GL(V)* is the group of all endomorphisms on a vector space V over a field F under addition and composition of maps. The special linear group SL(n,q) is the group of all invertible $n \times n$ linear matrices with determinant 1. The projective special linear group, PSL(n,q) is the quotient group of the special linear group and its center, that is

$$PSL(n, q) = SL(n, q)/Z(SL(n, q))$$

Given a set X, either finite or infinite, a transformation of X is a bijective self-map $f : X \to X$. Such a bijective self-map is called a permutation or an automorphism of X. The collection of all such transformations form a group under composition called the Automorphism group denoted as, $AutX = f : X \to X|f \textit{ is bijective}$. Let $(G, *)$ be a group and X any set. An action of a group G on X is a map: G × X → X, $(g, x) \to g.x$ which satisfies: $g_1.(g_2.x) = (g_1 * g_2).x$ for all $g_1, g_2 \in G$ and all x ∈ X and e,g:$x = x$ for all x ∈ X. If G has a subgroup H then there is a transitive group action of G on the set $(G \setminus H)$ of the right cosets of H by right multiplication. From this it follows that the study of transitivity is equivalent to that of coset spaces.

A permutation group G is transitive on Ω if for all $\alpha, \beta \in \Omega$, there exists an element g ∈ G such that the image α^g of α under g is equal to β. The regular representation of a group G is the permutation representation induced by the action of G on itself by left multiplication. The regular representation of a group G over a field $\mathbb{F}$ corresponds to viewing $\mathbb{F}$G as a module over itself with the usual left multiplication. Let $\mathbb{F}$ and $\mathbb{L}$ be finite fields, then, $\mathbb{L}$ is an extension field of F if and only if $\mathbb{F} \subseteq \mathbb{L}$ [11]. We denote the field extension by $\mathbb{L}/\mathbb{F}$. An irreducible $\mathbb{F}$G -module over a field $\mathbb{F}$ is said to be absolutely irreducible if it is irreducible for any extension field $\mathbb{L}/\mathbb{F}$. Any proper sub-module of a finitely generated module is contained in a maximal sub-module.

A linear code generated by a $k \times n$ generator matrix G is called an $[n, k]$ code. The elements of a code C are called the **code words** of the code. A linear code of dimension k contains precisely q^k code words. If all the code words are sequences of the same length n, then C is called a block code of length n. For a code C, the dual code $C^\perp = \{g|g \cdot x = 0 \ \forall x \in C\}$. A linear code C is self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$. Ternary codes are defined over the field $\mathbb{F}_3$. The **Hamming distance** of a code $d(u, v), u, v \in C$ is the number of positions in which the entries of two code words in a code differ. The weight of a codeword, denoted as $w(x)$ for $x \in C$ is the number of non-zero co-ordinates in the code words. The minimum weight is equal to the minimum distance among all non-zero codewords. If C is a linear code of length n over $\mathbb{F}_q^n$ then any isomorphism of C onto itself is called an automorphism of C. The set of all automorphisms of C is called **automorphism group of C**, denoted by $Aut(C)$.

If $\mathcal{D}$ is a $(v; k, \lambda)$ design, then the $b \times v$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1, & \text{if the i-th block contains j;} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

is called an incidence matrix of the design. A design is trivial if every $k - set$ of points is incident with a block of the design. It is called simple if distinct blocks are not incident with the same set of k points, self dual if it is isomorphic to its dual and symmetric, if $b = v$, i.e., its incidence matrix is a square matrix.

3 Construction and Preliminary Results

In this section, we present existing lemmas and theorems that we use in our work and discuss the methods of constructions used.

Theorem 3.1. *(see [12]) Any transitive action of a group G on a subgroup H is equivalent to the action of G by left multiplication on a coset space G=H.*

Theorem 3.2. *(See [13]) If F is a field and G a finite group, then there is a bijective correspondence between finitely generated FG -modules and representations of G on finite dimensional F -vector spaces.*

Theorem 3.3. *(Maschke's theorem) Let G be a finite group and $\mathbb{F}$ a field whose characteristic is 0 or a prime p that does not divide the order of G. Then every $\mathbb{F}G$ -module V is completely reducible i.e if V is an $\mathbb{F}G$-module and U any submodule of V, then there exists a submodule W of V such that $V = U \oplus W$. In particular, the algebra $\mathbb{F}G$ is semi-simple. (See Proof in [14])*

Theorem 3.4. *(Krull-Schmidt Theorem) Every module M can be written as M= $M_1 \oplus M_2$ ... $\oplus$...$\oplus M_n$ where M_i are indecomposable. Furthermore, if $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ and also $M = N_1 \oplus N_2 \oplus ... \oplus N_p$ where N_j are also indecomposable, then $n = p$ and the M_j are isomorphic to the N_j , in some order. (See Proof in [15])*

Lemma 3.5. *Let C be a code over $\mathbb{F}_3$, every codeword of C has weight divisible by 3 if and only if C is self orthogonal.*

Theorem 3.6. *(See [16] Theorem 5.2.5) Let $\mathcal{D}$ be a self-dual 1- design obtained by taking all the images under G of a non-trivial orbit $\triangle$ of the point stabilizer in any of G's primitive representations and on which G acts primitively on points and blocks, then the automorphism group of $\mathbb{D}$ contains G.*

Theorem 3.7. *(see [16]Theorem 5.2.6) If C is a linear code of length n of a symmetric $1-(v;k,\lambda)$ design $\mathcal{D}$ over a finite field $\mathbb{F}_q$, then the automorphism group of $\mathcal{D}$ is contained in the automorphism group of C.*

Lemma 3.8. *. Let G be a finite group and Ω a finite G-set, then the $\mathbb{F}_q$ G- submodules of F are precisely the G-invariant codes (i.e., G-invariant subspaces of F.) Proof. see [5]*

Lemma 3.9. *If C is a code, then C is invariant under the group G if and only if $G \subseteq AUT(C)$ Proof. See[17]*

From the Lemma 3.9 above, it follows that, suppose G is a group, given a representation of the group elements of the group G by permutations, we can work modulo 3 to obtain a representation of the group G on a vector space V over the field $\mathbb{F}_3$. The invariant subspaces are then all the ternary linear codes C for which G is a subgroup of $Aut(C)$. Let G be a permutation group of degree n and V the corresponding $\mathbb{F}_3$ permutation module, the sub-modules of V are called the G-invariant ternary codes.

3.1 Construction method 1

This method is described in [5]. Let G be a permutation group on a finite field $\mathbb{F}$, the G-invariant sub-modules of F can be regarded as linear codes in $\mathbb{F}$ (see Lemma 3.8).

Lemma 3.8 sets the baseline for this method where all the sub-modules of the permutation module are required. The decomposition of the permutation module over a field follows from Maschke's

Theorem in which case, it can be written as a sum of its irreducible sub-modules. An application of Krull-Schmidt's Theorem further shows that the module of a finite length can be written as a direct sum of indecomposable sub-modules and this decomposition is unique up to isomorphism and the order of the summands. In view of the mentioned results, this method was used in the study of the binary codes generated from $L_3(4)$ (see[5]). Further, using this method, all the binary codes from the permutation representations of the groups $PSU_4(2)$ and $PSU_3(3)$ obtained have been completely characterized by Brooke, (see[3]).

Given a permutation group G acting on a finite set Ω and ρ: G → GL(V) where ρ(g)(x) = g(x) with g ∈ G and x ∈ V , we follow the following steps: First we recognize $F_3\Omega$ as a permutation module and using Meat-Axe, we find all the maximal $F_3\Omega$-submodules. The non-isomorphic sub-modules are then used to determine the lattice diagrams of the permutation module. The sub-modules are the G-invariant codes and therefore, we check their isomorphic and equivalent copies. This method enables us to enumerate all the ternary linear codes obtained from this group. However, one of its shortcomings is that we are unable to find the minimum weights and automorphism groups of the codes with large dimensions.

3.2 Construction method 2

This method of constructing linear codes from groups is described in [18,19]. In this method we construct combinatorial designs from the orbits of the stabilizers of elements from the set that the group acts on. From the incidence matrices of these designs we are able to construct linear codes under various fields $\mathbb{F}_q$. This Method described by Key and Moori[18] is described by the Theorem and Lemma below.

Theorem 3.10. *1 Let G be a finite primitive permutation group acting on the set Ω of size n. Let $\alpha \in \Omega$ and let $\triangle \neq \alpha$ be an orbit of the stabilizer G_α of α. If $B = \triangle^g$: $g \in G$ and, given $\delta \in \triangle, \epsilon = \alpha, \delta^g : g \in G$, then $\mathbb{D} = (\oplus, \beta)$ forms a 1 - $(n, | \triangle |, | \triangle |)$ design with n blocks with G acting as an automorphism group primitive on points and blocks of the design.*

Proof. (see [18]). The group G acts transitively on X and hence it is a 1-design with [G : $G_\triangle$] blocks. Since △ is an orbit of G_x, $G_x \subseteq G_\triangle$ and given that G is primitive on X, we have G_x is maximal in G, hence $G_x = G_\triangle$, therefore, the number of blocks is n. Further, let $g, g' \in G$. Consider $\triangle^g$ and $\triangle^{g'}$ two blocks of $\mathbb{D}$, hence G is transitive on the blocks. If $\delta^g \in \triangle^g$, then $G \subseteq Aut(D)$ and the result follows. □

Lemma 3.11. *If the group G acts primitively on the points and the blocks of a symmetric 1-design $\mathcal{D}$, then the design can be obtained by orbiting a union of a point stabilizer.*

Proof. Suppose G acts primitively on points and blocks of the 1 -(v, k, k) design $\mathcal{D}$, let β be the block set of $\mathcal{D}$, then if β is any block of $\mathcal{D}$, $\beta = \beta^G$. Thus $|G| = |\beta||G_\beta|$ and since G is primitive, G_β is maximal and thus $G_\beta = G_\alpha$ for some point. Thus G_α fixes β, so this must be a union of orbits of G_α. □

We use the above 2 construction methods to enumerate the ternary linear codes and combinatoric designs from $L_3(4)$. We also discuss the comparison between the binary and ternary linear codes obtained from this group.

4 Ternary Linear Codes Invariant under $L_3(4)$

The group L_3 (4) is a non abelian finite simple group of order 20,160. It is the group of all 3 ×3 non singular matrices of dimension 1 over F_4 and a subgroup of 15 sporadic simple groups. It has a rich geometric structure that enables study of the interplay between the codes and combinatoric designs. Table 1 below represents the primitive representations as provided in the Atlas of Finite Groups (See [9]).

Table 1. Maximal Subgroups of $L_3(4)$

No.	Degree	Maximal Subgroups	No. of. orbits	Orbit length
1	21	2^4: A_5	2	1, 20
2	21	2^4: A_5	2	1, 20
3	56	A_6	3	1, 10, 45
4	56	A_6	3	1, 10, 45
5	56	A_6	3	1, 10, 45
6	120	$L_3(2)$	4	1, 21, 42, 56
7	120	$L_3(2)$	4	1, 21, 42, 56
8	120	$L_3(2)$	4	1, 21, 42, 56
9	280	$3^2 : Q_8$	8	1, 9, 18(3), 72(3)

4.1 Codes and designs using construction method 1

Using this method, we obtain the $\mathbb{F}_q$G-sub-modules of F which shall be the G-invariant codes and with the help of Meataxe, we compute the irreducible representations of $L_3(4)$. We enumerate all the non trivial ternary linear codes generated from the maximal sub-groups of dimensions 21, 56 and 120. Every conjugacy class of maximal sub groups of $L_3(4)$ contains a permutation module over the field GF3. From these permutation modules, we generate a chain of maximal sub modules recursively as every maximal sub-module represents a ternary linear code. This process terminates after obtaining an irreducible sub-module. We then determine the equivalence of the codes represented by these sub-modules since the dimensions of the sub-modules are those of our codes. Excluding the isomorphic copies, we enumerate and classify all the non trivial ternary linear codes invariant under $L_3(4)$.

4.2 A 21 dimensional representation

The group G acts primitively as a rank 2 group of degree 21 on each of the orbits $2^4 : A_5$ with orbits of lengths 1 and 20. The module of dimension 21 splits into two absolutely irreducible constituents of dimensions 1 and 19 over GF_3 with multiplicities 2 and 1 respectively. It has only one irreducible maximal sub-module of dimension 1. This is an absolutely irreducible sub module. The 21-dimensional module has one maximal sub- module of dimension 20. This 20-dimensional module has one maximal irreducible sub module of dimension 1.

Remark 4.1. The ternary linear codes obtained from MAGMA using these maximal sub-modules are all trivial unlike the 8 non-trivial binary codes generated from the same representation as seen in [8].

4.3 A 56 dimensional representation

As can be observed from table 4, the group G acts primitively as a rank 3 group of degree 56 on each of the orbits A_6 with orbits of lengths 1, 10 and 45. The permutation Module of dimension 56 has only two irreducible sub modules of dimensions 1 and 19. It splits into three absolutely irreducible constituents of dimensions 1, 15 and 19 with multiplicities 3, 1 and 2 respectively. We recursively find all maximal sub-modules of each module, terminating the process after obtaining an irreducible maximal sub-module. The 56-dimensional permutation module has two maximal sub modules of dimensions 37 and 55. The 55-dimensional module has one maximal sub module of dimension 36. The 37 dimensional module has 14 maximal sub-modules, one of dimension 22 and thirteen of dimension 36. From all the 36 dimensional sub modules, 5 of them are non isomorphic. Each of the five non-isomorphic 36-dimensional sub modules has five maximal sub modules; one of dimension 21 and four of dimension 35. From all the 35 dimensional sub modules, 5 of them are non isomorphic.

The module of dimension 22 has thirteen maximal sub modules, each of dimensions 21. From all the 21 dimensional sub modules, five are non isomorphic. Each of the 35 dimensional modules has two maximal sub modules, one of dimension 20 and another of dimension 34. All the 34 dimensional sub modules are isomorphic. Each of the five 21 dimensional modules has four 20 dimensional maximal sub modules from which five of them are non isomorphic. Of the five modules of dimension 20, four have one maximal sub module of dimension 19 which is irreducible. The remaining one has two irreducible maximal sub modules each, one of dimension 1 and the other of dimension 19. The module of dimension 34 has one irreducible maximal sub module of dimension 19. We therefore obtain 26 maximal sub modules from this permutation module. These are of dimensions;1, 19, 20, 20, 20, 20, 20, 21, 21, 21, 21, 21, 22, 34, 35, 35, 35, 35, 35, 36, 36, 36, 36, 36, 37 and 55. This information is represented in the lattice diagram below.

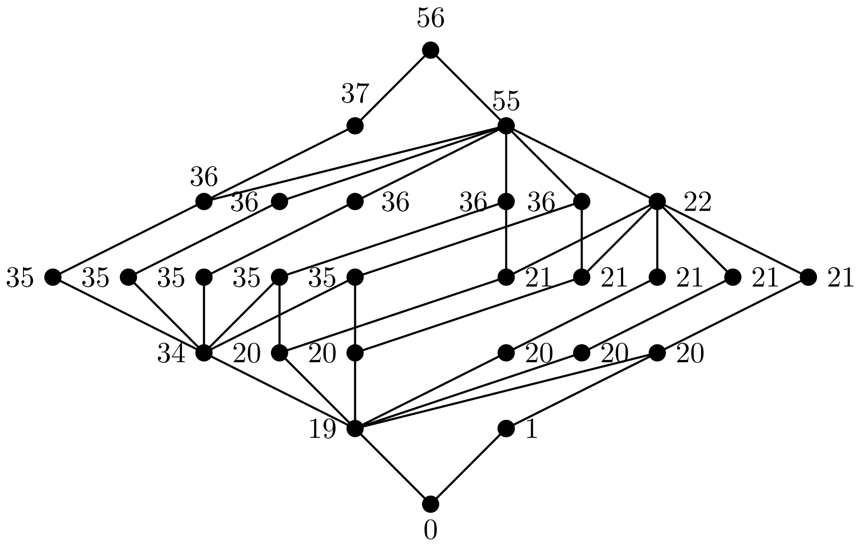

Fig. 1. 56-dimensional lattice diagram

4.4 Ternary linear codes from a 56 dimensional representation

From the sub-modules listed above, we obtain 16 non trivial ternary linear codes invariant under $L_3(4)$ and their respective duals. In the tables below, we indicate the codes generated, denoting them as $C_{56,i}$ and their duals as $C_{56,i}^{\perp}$ for $i = 1, 2, \cdots, 16$ respectively.

Table 2. Ternary linear codes of length 56

$C_{56,i}$ Code	[n,k,d] Code	Dual Codes	[n,k,d] Code
$C_{56,1}$	$[56, 19, 18]_3$	$C_{56,1}^{\perp}$	$[56, 37, 7]_3$
$C_{56,2}$	$[56, 20, 16]_3$	$C_{56,2}^{\perp}$	$[56, 36, 7]_3$
$C_{56,3}$	$[56, 20, 18]_3$	$C_{56,3}^{\perp}$	$[56, 36, 8]_3$
$C_{56,4}$	$[56, 20, 14]_3$	$C_{56,4}^{\perp}$	$[56, 36, 8]_3$
$C_{56,5}$	$[56, 20, 18]_3$	$C_{56,5}^{\perp}$	$[56, 36, 7]_3$
$C_{56,6}$	$[56, 20, 18]_3$	$C_{56,6}^{\perp}$	$[56, 36, 8]_3$
$C_{56,7}$	$[56, 20, 11]_3$	$C_{56,7}^{\perp}$	$[56, 36, 8]_3$
$C_{56,8}$	$[56, 20, 16]_3$	$C_{56,8}^{\perp}$	$[56, 36, 7]_3$
$C_{56,9}$	$[56, 21, 16]_3$	$C_{56,9}^{\perp}$	$[56, 35, 8]_3$
$C_{56,10}$	$[56, 21, 11]_3$	$C_{56,10}^{\perp}$	$[56, 35, 8]_3$
$C_{56,11}$	$[56, 21, 11]_3$	$C_{56,11}^{\perp}$	$[56, 35, 8]_3$
$C_{56,12}$	$[56, 21, 14]_3$	$C_{56,12}^{\perp}$	$[56, 35, 8]_3$
$C_{56,13}$	$[56, 21, 14]_3$	$C_{56,13}^{\perp}$	$[56, 35, 8]_3$
$C_{56,14}$	$[56, 21, 16]_3$	$C_{56,14}^{\perp}$	$[56, 35, 7]_3$
$C_{56,15}$	$[56, 21, 11]_3$	$C_{56,15}^{\perp}$	$[56, 35, 8]_3$
$C_{56,16}$	$[56, 22, 11]_3$	$C_{56,16}^{\perp}$	$[56, 34, 8]_3$

We also provide a partial listing of their weight distributions, their properties and the Automorphism group of every non-trivial ternary linear code invariant under L3(4) in the tables below.

Table 3. Weight distributions of the codes from a 56-dimensional representation

Name	dim	0	7	8	10	11	12	13	14	15	16	17	18	19	20	21	22
$C_{56,1}$	19	1											4200			8064	
$C_{56,2}$	20	1									1806		4200	3360		8064	
$C_{56,3}$	20	1											4200			26208	
$C_{56,4}$	20	1											4200		29736		
$C_{56,5}$	20	1											4200		11592	8064	
$C_{56,6}$	20	1											4200			37296	
$C_{56,7}$	20	1				112							4200		16744	8064	
$C_{56,8}$	20	1									630		4200			8064	161280
$C_{56,9}$	21	1									1806		4200	3360	11592	55440	114240
$C_{56,10}$	21	1				112			240		3612		4200	6720	46480	8064	28480
$C_{56,11}$	21	1				112							4200		39928	26208	
$C_{56,12}$	21	1							240				4200		52920	910560	
$C_{56,13}$	21	1							240		630		4200		29736	44352	161280
$C_{56,14}$	21	1									2436		4200	3360	23184	8064	275520
$C_{56,15}$	21	1				112					630		4200		16744	66528	161280
$C_{56,16}$	22	1				112			240		4242		4200	6720	92848	102816	389760

Table 4. Continuation of table 3

Name	23	24	25	26	27	28	29	30	31	32
$C_{56,1}$		528150			6309520			48544272		
$C_{56,2}$		528150	2424576		6309520	27549120		48544272	159617808	
$C_{56,3}$		1631910			19141360		144435312			
$C_{56,4}$	194880	528150		6477072	6309520		51794400	48544272		250830090
$C_{56,5}$	357840	528150		5942160	6309520		52743600	48544272		249512130
$C_{56,6}$		1515990			19690720			142877952		
$C_{56,7}$	312480	528150		5962992	6309520		53245360	48544272		247932090
$C_{56,8}$		528150	2306304		6309520	27740880		48544272	158911200	
$C_{56,9}$	357840	2619750	2424576	5942160	32522560	27549120	52743600	238768992	159617808	249512130
$C_{56,10}$	507360	528150	4849152	12440064	6309520	55098240	105039760	48544272	319235616	498762180
$C_{56,11}$	1028160	1631910		17847312	19141360		158732560	144435312		746956350
$C_{56,12}$	910560	1515990		18361392	19690720		157281600	142877952		749854350
$C_{56,13}$	194880	2735670	2306304	6477072	31973200	27740880	51794400	240326352	158911200	250830090
$C_{56,14}$	715680	528150	4730880	11884320	6309520	55290000	105487200	48544272	318529008	499024260
$C_{56,15}$	312480	2503830	2306304	5962992	33071920	27740880	53245360	237211632	158911200	247932090
$C_{56,16}$	1938720	4711350	7155456	36208704	58735600	82839120	316014160	428993712	478146816	1496810700

Table 5. Continuation of table 4

Name	33	34	35	36	37	38	39	40	41
$C_{56,1}$	179524800			361395468			357966000		
$C_{56,2}$	179524800	489444480		361395468	779390640		357966000	608968080	
$C_{56,3}$	540499680			1083265092			1073021040		
$C_{56,4}$	179524800		614142144	361395468		781089120	357966000		472956960
$C_{56,5}$	179524800		616295952	361395468		777712320	357966000		476555184
$C_{56,6}$	543432960			1079412516					
$C_{56,7}$	179524800		618365808	361395468		776577200	357966000		476449344
$C_{56,8}$	179524800	491717520		361395468	775383840		357966000	612930024	
$C_{56,9}$	904407840	489444480	616295952	1801282140	779390640	777712320	1791674640	608968080	476555184
$C_{56,10}$	179524800	978888960	1232507952	361395468	1558781280	1557666320	357966000	1217936160	949406304
$C_{56,11}$	540499680		540499680	1083265092		2332001840	1073021040		1429559712
$C_{56,12}$	543432960		1846734048	1079412516		2336513760	1076619600		1426067328
$C_{56,13}$	901474560	491717520	614142144	1805134716	775383840	781089120	1788076080	612930024	472956960
$C_{56,14}$	179524800	981162000	1232591904	361395468	1554774480	1555424640	357966000	1221898104	953110368
$C_{56,15}$	907341120	491717520	618365808	1797429564	775383840	776577200	1795273200	612930024	476449344
$C_{56,16}$	1629290880	1470606480	3697691760	3241168812	2334165120	4668515600	3225383280	1830866184	2855627040

Table 6. Continuation of table 5

Name	42	43	44	45	46	47	48	49	50	51	52
$C_{56,1}$	170451360			34736016			2769480			18816	
$C_{56,2}$	170451360	221798640		34736016	33536832		2769480	1633200		18816	
$C_{56,3}$	512718480			103568304			8377110			85344	
$C_{56,4}$	170451360		132879600	34736016		13547520	2769480		577248	18816	
$C_{56,5}$	170451360		130591440	34736016		14323680	2769480		476784	18816	
$C_{56,6}$	510329520			104672064			8039430			146832	
$C_{56,7}$	170451360		130974480	34736016	14241920	2769480		425376	18816		
$C_{56,8}$	170451360	219653280		34736016	34047216		2769480	1653120		18816	17640
$C_{56,9}$	852596640	221798640	130591440	173504352	33536832	14323680	13647060	1633200	476784	213360	39480
$C_{56,10}$	170451360	443597280	263854080	34736016	67073664	27789440	2769480	3266400	1002624	18816	78960
$C_{56,11}$	512718480		392157360	103568304		42889280	8377110		1378944	85344	
$C_{56,12}$	510329520		394062480	104672064		42194880	8039430	1530816	146832		
$C_{56,13}$	854985600	219653280	132879600	172400592	34047216	13547520	13984740	1653120	577248	151872	17640
$C_{56,14}$	170451360	441451920	261182880	34736016	67584048	28647360	2769480	3286320	953568	18816	57120
$C_{56,15}$	850207680	219653280	130974480	174608112	34047216	14241920	13309380	1653120	425376	274848	17640
$C_{56,16}$	1534741920	663250560	786219840	312272688	101120880	85084160	24524640	4919520	2909760	407904	96600

Table 7. Continuation of table 6

Name	53	54	55	56
$C_{56,1}$		5320		
$C_{56,2}$		5320	672	
$C_{56,3}$		10360		
$C_{56,4}$	3360	5320		564
$C_{56,5}$		5320		252
$C_{56,6}$		5320		
$C_{56,7}$	18480	5320		548
$C_{56,8}$		5340		
$C_{56,9}$		10360	672	252
$C_{56,10}$	21840	5320	1344	1112
$C_{56,11}$	18480	10360		1052
$C_{56,12}$	3360	5320		1068
$C_{56,13}$	3360	15400		564
$C_{56,14}$		5320	672	504
$C_{56,15}$	18480	5320		548
$C_{56,16}$	21840	15400	1344	2120

Table 8. Weight distributions of the dual codes from a 56-dimensional representation

Name	dim	0	7	8	10	11	12	13	14	15
$C^{\perp}_{56,16}$	34	1		630	13104		206640	362880	3679200	16813440
$C^{\perp}_{56,15}$	35	1		3150	29232	40320	430920	1370880	10827360	52069248
$C^{\perp}_{56,14}$	35	1		630	13104	2352	433440	992880	9388800	50652000
$C^{\perp}_{56,13}$	35	1		630	13104	28224	246960	1078560	10257120	49815360
$C^{\perp}_{56,12}$	35	1		630	13104	10080	378000	1139040	8830080	50790096
$C^{\perp}_{56,11}$	35	1		630	23184	13440	37290	1824480	10315440	52043040
$C^{\perp}_{56,10}$	35	1		630	17136		549360	1411200	10231200	52657920
$C^{\perp}_{56,9}$	35	1	240	630	14784	5040	315000	1370880	9046800	49322448
$C^{\perp}_{56,8}$	36	1		3150	29232	99120	738360	3432240	29692800	151911648
$C^{\perp}_{56,7}$	36	1		630	23184	61824	756000	4092480	2795120	152998272
$C^{\perp}_{56,6}$	36	1		630	17136	22512	1118880	3593520	26242560	154449792
$C^{\perp}_{56,5}$	36	1	240	630	18816	43344	869400	3911040	27327600	152145504
$C^{\perp}_{56,4}$	36	1	240	3150	30912	65520	882000	3931200	26496720	152531568
$C^{\perp}_{56,3}$	36	1	480	630	26544	25872	816480	4470480	267602240	150899616
$C^{\perp}_{56,2}$	36	1		3150	47376	563760	1282680	4929120	30567600	158987808
$C^{\perp}_{56,1}$	37	1	480	3150	50736	162980	2492280	12111120	80771760	459754848

Table 9. Continuation of table 8

Name	16	17	18	19	20	21
$C^{\perp}_{56,16}$	90852300	403945920	1785733320	7067128320	26258710212	89888334624
$C^{\perp}_{56,15}$	268104690	1220546880	5335324680	21227330880	78794205588	269657119008
$C^{\perp}_{56,14}$	262607436	1226504160	5313471240	21281027040	78784631436	270051593184
$C^{\perp}_{56,13}$	264383532	1214897040	5334105000	21265735680	78806918484	269900138160
$C^{\perp}_{56,12}$	262563084	1228132080	5324967480	21276712800	78729719292	270096257376
$C^{\perp}_{56,11}$	266644476	1231826400	5325385800	21229115040	78692697804	269898554592
$C^{\perp}_{56,10}$	269917452	1226252160	5323521000	21220476480	78720552036	269796024864
$C^{\perp}_{56,9}$	263806746	1224447840	5333817720	21279012720	78767847396	269860159152
$C^{\perp}_{56,8}$	786922290	3665007360	15959805960	63838444320	236416543356	809843984640
$C^{\perp}_{56,7}$	783597276	3691149840	15952225800	63846891360	236182924236	810326203632
$C^{\perp}_{56,6}$	785094156	3697182720	15929727240	63853544160	236188491420	810375128928
$C^{\perp}_{56,5}$	788113914	3681891360	15959211240	63840552720	236248906572	809987575680
$C^{\perp}_{56,4}$	784480704	3689421120	15961877400	63858684240	236245360932	810044789040
$C^{\perp}_{56,3}$	784308504	3695388480	15949292520	63866782560	236236893396	810005462208
$C^{\perp}_{56,2}$	802027170	3693039840	15950552520	63696013920	236151876828	809482719456
$C^{\perp}_{56,1}$	2353596798	11075248800	47828139240	191569234080	708676525284	2430444925152

Table 10. Properties of the codes and their duals

$C_{56,i}$	Self Orthogonal	Self Dual	Aut($C_{56,i}$) Order	Aut($C_{56,i}$)	$C^{\perp}_{56,i}$	Self Orthogonal	Self Dual	Aut($C^{\perp}_{56,i}$) Order	Aut($C^{\perp}_{56,i}$)
$C_{56,1}$	Yes	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,1}$	No	No	161280	$L_3(4):2^3$
$C_{56,2}$	No	No	80640	$L_3(4):2^2$	$C^{\perp}_{56,2}$	No	No	80640	$L_3(4):2^2$
$C_{56,3}$	Yes	No	80640	$L_3(4):2^2$	$C^{\perp}_{56,3}$	No	No	80640	$L_3(4):2^2$
$C_{56,4}$	No	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,4}$	No	No	161280	$L_3(4):2^3$
$C_{56,5}$	No	No	40320	$L_3(4):2$	$C^{\perp}_{56,5}$	No	No	40320	$L_3(4):2$
$C_{56,6}$	Yes	No	80640	$L_3(4):2^2$	$C^{\perp}_{56,6}$	No	No	80640	$L_3(4):2^2$
$C_{56,7}$	No	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,7}$	No	No	161280	$L_3(4):2^3$
$C_{56,8}$	No	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,8}$	No	No	161280	$L_3(4):2^3$
$C_{56,9}$	No	No	40320	$L_3(4):2$	$C^{\perp}_{56,9}$	No	No	40320	$L_3(4):2$
$C_{56,10}$	No	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,10}$	No	No	161280	$L_3(4):2^3$
$C_{56,11}$	No	No	80640	$L_3(4):2^2$	$C^{\perp}_{56,11}$	No	No	80640	$L_3(4):2^2$
$C_{56,12}$	No	No	80640	$L_3(4):2^2$	$C^{\perp}_{56,12}$	No	No	80640	$L_3(4):2^2$
$C_{56,13}$	No	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,13}$	No	No	161280	$L_3(4):2^3$
$C_{56,14}$	No	No	80640	$L_3(4):2^2$	$C^{\perp}_{56,14}$	No	No	80640	$L_3(4):2^2$
$C_{56,15}$	No	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,15}$	No	No	161280	$L_3(4):2^3$
$C_{56,16}$	No	No	161280	$L_3(4):2^3$	$C^{\perp}_{56,16}$	No	No	161280	$L_3(4):2^3$

Remark 4.2. From the results in the tables above, we make the following deductions;

(i) There are no self-dual ternary codes invariant under $L_3(4)$ as is the case in binary codes.

(ii) $L_3(4)$ is not realized as a full Automorphism group of any code.

(iii) $C_{56,1} \subset C_{56,7} \subset C_{56,11} \subset C_{56,16}$, $C_{56,1} \subset C_{56,8} C_{56,1} \subset C_{56,7} \subset C_{56,10} \subset C_{56,15}$.

(iv) $C_{56,3} \subset C_{56,11}$, $C_{56,2} \subset C_{56,14}$ and $C_{56,6} \subset C_{56,12}$.

(v) $C_{56,5} \subset C_{56,9}$

Our results are summarized below.

Theorem 4.1. *Let $C_{56,i}$ denote the ternary codes for i = 1, 2, ..., 16 and $C^{\perp}_{56,i}$ their respective duals for $i = 1, 2, \cdots, 16$. Then,*

(i) The linear ternary code $C_{56,1}$ is a $[56, 19, 18]_3$ self-orthogonal irreducible code. It is not optimal and is at a distance of 3 from optimal. Its dual is the $[56, 37, 7]_3$ code.

(ii) $C_{56,3}$ is a $[56, 20, 18]_3$ self -orthogonal code whose automorphism group is $L_3(4) : 2^2$. It is not isomorphic to $C_{56,6}$ that has the same properties.

(iii) The Automorphism group of the codes $C_{56,i}$ for i := 1, 4, 7, 8, 10, 13, 15 and 16 is $L_3(4).2^3$.

(iv) The Automorphism group of the codes $C_{56,i}$ for i := 2, 3, 6, 11, 12 and 14 is $L_3(4).2^2$.

(v) The Automorphism group of codes $C_{56,i}$ for i := 5 and 9 is $L_3(4).2$.

Proof. **Case (i)** The code words of the code $[56, 19, 18]_3$ have weights all divisible by 3 as can be observed in tables $3, \cdots, 7$. It ia also clear from the weight distribution tables above that the weights of these code words are divisible by 3. But, it well known(cf.[13]) that a ternary code is self orthogonal if the weight of all the code words are divisible by 3. It follows that $[56, 19, 18]_3$ is self orthogonal. From the Grassl tables, the optimal ternary linear code of length 56 and dimension 19 has a minimum weight of 21 (see [20]). From this, we conclude that $[56, 19, 18]_3$ is a not optimal and is at a distance of 3 from the optimal. The code $[56, 19, 18]_3$ is represented by the maximal sub-module of dimension 19. This maximal sub-module is irreducible as it decomposes into the trivial 1 and 0 dimensional sub-modules. The code $[56, 19, 18]_3$ is therefore irreducible.

Case (ii) The linear code $C_{56,3}$ has code words with weights all divisible by 3 from the weight distribution provided above. Similarly, it follows that $C_{56,3}$ is self orthogonal. The Automorphism Group of $C_{56,3}$ has order $80640 = 2^2 \times 20160$. The composition factors of this automorphism group are two cyclic groups of order 2 and a primitive group of order 20160. The code $C_{56,3}$ is invariant under $L_3(4)$ and hence by a result in [8], $L_3(4) \subseteq Aut(C_{56,3})$. On the other hand, the code words of $C_{56,6}$ have weights all divisible by 3 hence $C_{56,6}$ is also self orthogonal. The Automorphism Group of $C_{56,6}$ has order $80640 = 2^2\times$ 20160. The composition factors of this automorphism group are two cyclic groups of order 2 and a primitive group of order 20160, similar to that of $C_{56,3}$. The code $C_{56,6}$ is invariant under $L_3(4)$ and hence $L_3(4) \subseteq Aut(C_{56,6})$. We thus conclude that the $Aut(C_{56,3}) = Aut(C_{56,6}) \cong L_3(4).2^2$.

Case (iii) From table 10, the order of the automorphism groups of the codes $C_{56,i}$ for $i = 1, 7, 8, 10, 13, 15$ and 16 is 161,280. We obtained the composition factors of this group of order 161,280 to be three cyclic subgroups of order 2 and a primitive group of order 20160 from MAGAMA. Since these codes are all invariant under $L_3(4)$, it follows that $L_3(4) \subseteq Aut(C_{56,i})$ for i = 1, 7, 8, 10, 13, 15 and 16. Therefore, $Aut(C_{56,i}) = L_3(4).2^3$ for the cited cases.

Case (iv) Let G be the automorphism group of the codes $C_{56,i}$ for i := 2, 3, 6, 11, 12 and 14. Computations from magma show that the automorphism groups of these codes have an order of 161280 and are isomorphic. As proved in part (iii) above, the composition factors of this automorphism group are two cyclic groups of order 2 and a primitive group of order 20160. These codes are invariant under $L_3(4)$ and hence $L_3(4) \subseteq Aut(C_{56,i})$. We thus conclude that the automorphism group is $L_3(4).2^2$.

Case (v) Follows a similar fashion used in proofs of parts (iii) and (iv) with modifications. Thus, $Aut(C_{56,i}) = L_3(4).2$ for i=5,9. □

4.5 Ternary linear codes of length 120 invariant under $L_3(4)$

From table 1 and the Atlas of Finite groups, the group L3(4) acts primitively as a rank 4 group of degree 120 on each of the orbits $L_3(2)$ of lengths 1, 21, 42 and 56. The 120 dimensional permutation Module splits into three absolutely irreducible constituents of dimensions 1, 15, 15, 15 and 19 with multiplicities 3, 2, 1, 1 and 3 respectively. It has only two irreducible sub modules of dimensions 1 and 15. These are absolutely irreducible sub modules. The 120-dimensional permutation module has two maximal sub modules of dimensions 105 and 119. The 105 dimensional module has 2 maximal sub modules, one of dimension 86 and another of dimension 104. The 119-dimensional module has two maximal sub modules one of dimension 100 and the other of dimension 104. The 104 dimensional sub modules are isomorphic. The 86 dimensional module has 6 maximal sub-modules, the 100 dimensional module has 2 maximal sub-modules while the 104 dimensional module has 4 maximal sub-modules. From these sub-modules, we have two non-isomorphic 71 dimensional ones, 4 pairs of 85 dimensional isomorphic sub-modules, and one 99 dimensional sub-module. Working recursively, we get 67 non isomorphic sub modules of the dimensions as follows; 1, 15, 16, 20, 21, 34, 35(7), 36(2), 49(2), 50(8),51(2), 54, 55(4), 56, 64, 65(4), 66, 69(2), 70(8), 71(2), 84(2), 85(7), 86, 99, 100, 104, 105 and 119. From these maximal sub-modules, the non trivial codes obtained are of dimensions 15, 16, 20, 21, 34, 35, 35, 35, 35, 35, 35, 35, 36, 36, 49, 49, 50, 50, 50, 50, 50, 50, 50, 50, 51, 51, 54, 55, 55, 55, 55, 56 and their respective duals. Figure 2 below illustrates the lattice diagram of the maximal sub-modules.

From the maximal sub-modules, we obtained 32 non trivial ternary linear codes up to isomorphism invariant under $L_3(4)$ as well as their respective duals. In the tables below, we indicate the codes generated, denoting them as $C_{120,i}$ and their duals as $C_{120,i}^{\perp}$ for $i = 1, 2, \cdots, 32$ respectively. Due to the large dimensions of some of these codes, we did not manage to find the minimum distance of all the codes. We provide a partial listing of their weight distributions, and the properties obtained.

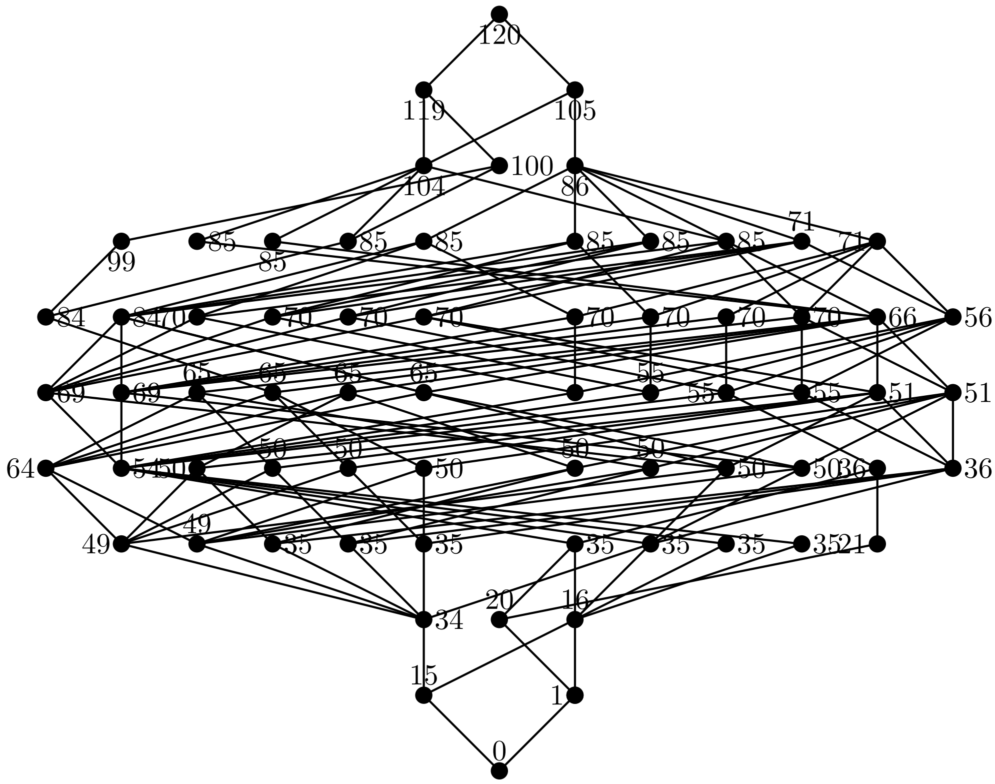

Fig. 2. 120 degree subgroup lattice diagram

Table 11. Codes of dimension 120 and their duals

$C_{120,i}$	$C_{120,i}^{\perp}$	$C_{120,i}$	$C_{120,i}^{\perp}$
$C_{120,1}$ $[120,15,48]_3$	$C_{120,1}^{\perp}$ $[120,105]_3$	$C_{120,17}$ $[120,50]_3$	$C_{120,17}^{\perp}$ $[120,70]_3$
$C_{120,2}$ $[120,16,48]_3$	$C_{120,2}^{\perp}$ $[120,104]_3$	$C_{120,18}$ $[120,50,16]_3$	$C_{120,18}^{\perp}$ $[120,70]_3$
$C_{120,3}$ $[120,20,30]_3$	$C_{120,3}^{\perp}$ $[120,100,6]_3$	$C_{120,19}$ $[120,50]_3$	$C_{120,19}^{\perp}$ $[120,70]_3$
$C_{120,4}$ $[120,21,30]_3$	$C_{120,4}^{\perp}$ $[120,99,6]_3$	$C_{120,20}$ $[120,50]_3$	$C_{120,20}^{\perp}$ $[120,70]_3$
$C_{120,5}$ $[120,34,24]_3$	$C_{120,5}^{\perp}$ $[120,86]_3$	$C_{120,21}$ $[120,50]_3$	$C_{120,21}^{\perp}$ $[120,70]_3$
$C_{120,6}$ $[120,35,30]_3$	$C_{120,6}^{\perp}$ $[120,85]_3$	$C_{120,22}$ $[120,50]_3$	$C_{120,22}^{\perp}$ $[120,70]_3$
$C_{120,7}$ $[120,35,15]_3$	$C_{120,7}^{\perp}$ $[120,85]_3$	$C_{120,23}$ $[120,50]_3$	$C_{120,23}^{\perp}$ $[120,70]_3$
$C_{120,8}$ $[120,35,15]_3$	$C_{120,8}^{\perp}$ $[120,85]_3$	$C_{120,24}$ $[120,50]_3$	$C_{120,24}^{\perp}$ $[120,70]_3$
$C_{120,9}$ $[120,35,24]_3$	$C_{120,9}^{\perp}$ $[120,85]_3$	$C_{120,25}$ $[120,51,16]_3$	$C_{120,25}^{\perp}$ $[120,69]_3$
$C_{120,10}$ $[120,35,24]_3$	$C_{120,10}^{\perp}$ $[120,85]_3$	$C_{120,26}$ $[120,51,16]_3$	$C_{120,26}^{\perp}$ $[120,69]_3$
$C_{120,11}$ $[120,35,16]_3$	$C_{120,11}^{\perp}$ $[120,85]_3$	$C_{120,27}$ $[120,54,15]_3$	$C_{120,27}^{\perp}$ $[120,66]_3$
$C_{120,12}$ $[120,35,24]_3$	$C_{120,12}^{\perp}$ $[120,84]_3$	$C_{120,28}$ $[120,55,15]_3$	$C_{120,28}^{\perp}$ $[120,65]_3$
$C_{120,13}$ $[120,36,22]_3$	$C_{120,13}^{\perp}$ $[120,84]_3$	$C_{120,29}$ $[120,55,15]_3$	$C_{120,29}^{\perp}$ $[120,65]_3$
$C_{120,14}$ $[120,36,16]_3$	$C_{120,14}^{\perp}$ $[120,71]_3$	$C_{120,30}$ $[120,55,15]_3$	$C_{120,30}^{\perp}$ $[120,65]_3$
$C_{120,15}$ $[120,49]_3$	$C_{120,15}^{\perp}$ $[120,71]_3$	$C_{120,31}$ $[120,55,15]_3$	$C_{120,31}^{\perp}$ $[120,65]_3$
$C_{120,16}$ $[120,49]_3$	$C_{120,16}^{\perp}$ $[120,70]_3$	$C_{120,32}$ $[120,56,15]_3$	$C_{120,32}^{\perp}$ $[120,64]_3$

Table 12. Weight distributions of some codes from a 120-dimensional representation

Name	dim	0	30	32	36	42	43	44	46	47	48	49	50
$C_{120,1}$	15	1									1260		
$C_{120,2}$	16	1									1260		
$C_{120,3}$	20	1	112	210	560	1120		3360	5040	6720	1260		13272
$C_{120,4}$	21	1	112	210	1040	2772	1120	5400	10080	6720	1400	26880	14616

Table 13. Continuation of table 12

dim	51	52	53	54	55	56	57	58	59	60
15				2240						26208
16			2240			11520			43176	
20		25200	6720	55400	60480	78360	94080	156240	107520	428680
21	3360	31920	10080	73920	149184	280170	399840	287280	823200	912856

Table 14. Continuation of table 13

dim	62	63	64	65	66	67	68	69
15		240			108360			362880
16		26160			300440			1095360
20	1182720	815360	2313360	2287488	6731760	8442720	19092780	29302560
21	2919840	3973760	5901000	6621216	20017200	33201840	53337900	86382240

Remark 4.3. From the tables above, we make the following observations;

(i)We only found the Automorphism Groups of the codes $C_{120,1}$, $C_{120,3}$ $C_{120,4}$ and $C_{120,12}$ due to the large sizes of the codes.

(ii)$C_{120,1} \subset C_{120,3} \subset C_{120,4}$.

(iii) $L_3(4)$ is not realized as a full Automorphism group of the codes . It is however a subgroup of all the Automorphism groups of these codes.

We obtained no optimal linear ternary codes. We observe this by comparing the minimum weights of the codes obtained to the Grassl table for codes of length 120 and dimensions 15, 16, 20, 21, 34, 35, 36, 49, 50, 51, 55 and 56 respectively.

Theorem 4.2. *Let $C_{120,i}$ denote the ternary codes for $i = 1, 2, \cdots, 32$ and $C_{120,i}^{\perp}$, their respective duals, then;*

(i)The linear ternary code $C_{120,i}$, $[120, 15, 48]_3$ is irreducible, self-orthogonal and not optimal. The automorphism group of this code is $L_3(4) \cdot 2^3$.

(ii)The ternary linear codes $C_{120,3}$ and $C_{120,4}$ are self -orthogonal codes whose automorphism group is isomorphic to $L_3(4) \cdot 2^3$.

Proof. : **Case (i)**. The $[120, 15, 48]_3$ code is the maximal submodule of dimension 15. The composition factors of this submodule are trivial, of dimensions 1 and 0 as can be observed from the lattice figure 2. Therefore, this code is irreducible. From the weight distribution which is partially listed, the weights of all the codewords of $[120, 15, 48]_3$ are divisible by 3 and thus self-orthogonal. Also, from the Grassl tables we identify that $[56, 19, 18]_3$ is not optimal.

Case (ii). From the weight distribution of the ternary linear code $C_{120,3}$, all the weights of its codewords are divisible by 3. As in the proof of **Case (i)**, above, it follows that this code is self orthogonal. MAGMA computations show that the Automorphism group of this codes is of order $161280 = 2^3 \times 20160$. Up to isomorphism, this group decomposes into a primitive subgroup of order 20160 and three cyclic subgroups of order 2. The order of $L_3(4)$ divides the order of the automorphism group of $C_{120,3}$. We now prove that the group of order 20160 is $L_3(4)$. The code $C_{120,3}$ is invariant under $L_3(4)$ and thus $L_3(4)$ is a subset of the Automorphism group. Similarly, the weights of the codewords of $C_{120,4}$, are all divisible by 3 so this code is self orthogonal. Its automorphism group is also of order $161280 = 2^3 \times 20160$. Up to isomorphism, this group also decomposes into a primitive subgroup of order 20160 and three cyclic subgroups of order 2. The code $C_{120,4}$ is invariant under $L_3(4)$ and thus $L_3(4)$ is a subset of the Automorphism group. We can then deduce that$Aut(C56,3) \cong Aut(C_{56,4}) \cong L_3(4) \cdot 2^3$. □

5 Codes and Designs by Construction Method 2

We used the method derived by Key and Moori [18] to generate ternary linear codes from the 21, 56 and 120 -dimensional representations of $L_3(4)$. Using this method, we generate $1-(n, |\triangle|, |\triangle|)$ symmetric designs for n = 21,56 and 120 from the orbits of the stabilizers of the elements of the subgroups. Further, we obtain the union of these stabilizers and generate designs from their orbits. For a design $\mathcal{D}$ and the prime 3, the ternary code of the design is the code over F_3 generated by the rows of the incidence matrix. From this, we construct the ternary linear codes and discuss their Automorphism groups. The results are presented in the table below.

Table 15. Designs and the codes generated

Length of orbits	$t-(v,k,\lambda)$ Design	$[n,k,d]_3 Code$	Code as Rep.	Aut(Des)	Aut(Code)
10	1-(56,10,10)	$[56,56,1]_3$	Trivial code	$L_3(4).2^2$	
11	2-(56,11,2)	$[56,20,11]_3$	$C_{56,7}$	$L_3(4).2^2$	$L_3(4).2^3$
21	1-(120,21,21)	$[120,99,6]_3$	$C^{\perp}_{120,4}$	$L_3(4).2^2$	
22	1-(120,22,22)	$[120,36,22]_3$	$C_{120,113}$	$L_3(4).2^2$	
42	1-(120,42,42)	$[120,99,6]_3$	$C^{\perp}_{120,4}$	$L_3(4).2^2$	
43	1-(120,43,43)	$[120,21,30]_3$	$C_{120,4}$	$L_3(4).2^2$	
45	1-(56,45,45)	$[56,19,18]_3$	$C_{56,1}$	$L_3(4).2^2$	$L_3(4).2^3$
46	1-(56,46,46)	$[56,56,1]_3$	Trivial code	$L_3(4).2^2$	
56	1-(120,56,56)	$[120,120,1]_3$	Trivial code	$L_3(4).2^2$	
57	1-(120,57,57)	$[120,100,6]_3$	$C^{\perp}_{120,3}$	$L_3(4).2^2$	
63	1-(120,63,63)	$[120,99,6]_3$	$C^{\perp}_{120,4}$	$L_3(4).2^2$	
77	1-(120,77,77)	$[120,21,30]_3$	$C_{120,4}$	$L_3(4).2^2$	$L_3(4).2^3$
98	1-(120,98,98)	$[120,36,22]_3$	$C_{120,13}$	$L_3(4).2^2$	

Using this method, we were able to construct three trivial ternary linear codes. From table 1, we obtain the lengths of the orbits of the stabilizers of the elements of the subgroups. These are; 1, 10, 20, 21, 42, 45 and 56. We obtain the designs constructed from these orbits. Further, the unions of these orbits generate orbits of lengths; 11, 22, 43, 46, 57, 63, 77 and 98. All the designs from these orbits are listed in the table above. The table also contains the codes constructed from the incidence matrices of these designs.

Remark 5.1. From table 15 above, we make the following observations;

(i)The trivial codes $[21,21,1]_3$, $[56,56,1]_3$, $[56, 56, 1]_3$ and $[120, 120, 1]_3$ are constructed from the incidence matrices of the designs 1 - (21, 20, 20), 1 - (56, 10, 10), 1 - (56, 46, 46) and 1 -(120, 56, 56) respectively.

(ii) We obtained 10 non trivial codes using this method, all previously generated using the first method of construction.

(iii)The unique 2 - (56, 11, 2) self-dual symmetric design as found in [8], was also obtained using this method.

(iv) Isomorphic linear ternary codes such as $[120, 36, 22]_3$ were constructed from the incidence matrices of the designs 1 - (120, 22, 22) and 1 - (120, 98, 98).

Theorem 5.1. *The $[120, 99, 6]_3$ ternary linear code represented as the $C^{\perp}_{120,4}$ code in table 15 was constructed from the incidence matrix of the 1 - (120, 21, 21) and 1 - (120, 42, 42) designs. Its dual code is $[120, 21, 30]_3$ with the automorphism group isomorphic to $L_3(4).2^3$.*

Proof. From the four orbits of lengths 1, 21, 42 and 56 we generate the 1 - (120, 1, 1), 1 - (120, 21, 21), 1 - (120, 42, 42) and 1 - (120, 56, 56) designs respectively. From the incidence matrices of these designs, we were able to generate ternary linear codes and their duals. The first and last designs generated trivial codes while the second and forth generated the ternary linear code $[120, 99, 6]_3$. This code has an automorphism group of order 161280. From our magma computations, we were able to find the dual of this code which is the $[120, 21, 30]_3$ code with an automorphism group of order 161280. The composition factors of this automorphism group are two cyclic groups of order 2 and a primitive group of order 20160. There are two primitive groups of order 20160 which are A_8 and $L_3(4)$. These codes are invariant under $L_3(4)$ which leads to the deduction that $L_3(4)$ is a subset of the automorphism group. Further calculations from MAGMA show that the automorphism group of the design is of order 80640. So, it holds that $G \subset Aut(Des)$, $Aut(Des) \subset Aut(C_{56,6})$. Therefore, we can deduce that $L_3(4) \subset Aut(C_{56,6})$. It therefore holds that $Aut(C_{56,6}) \subset L_3(4).2^3$. □

Theorem 5.2. *The self-orthogonal non trivial ternary linear code $C_{56,1}$ with minimum weight 18 is constructed from the incidence matrix of the 1 - (56, 45, 45) self -dual symmetric design. $C_{56,1}$ is the $[56, 19, 18]_3$ with the automorphism group $Aut(C_{56,1}) \sim= L_3(4) : 2^3$. The automorphism group of the design is $Aut(Des) \cong L_3(4) : 2^2$*

Proof. : The automorphism group of $C_{56,1}$ has an order of $80640 = 2^2 \times 20160$. The composition factors of $Aut(C_{56,1})$ are three cyclic groups of order 2 and a primitive group of order 20160. There are two primitive groups of order 20160 which are A_8 and $L_3(4)$. This code is invariant under $L_3(4)$ which leads to the deduction that $L_3(4)$ is a subset of its automorphism group. Therefore, $Aut(C_{56,1}) = L_3(4).2^2$. Let $Aut(Des)$ denote the automorphism group of the 1 - (56, 45, 45) design that was obtained from an orbit of length 45. From our MAGAMA computations, we found that the order of $Aut(Des)$ is 80640. But it follows that $G \subset Aut(Des)$ and $Aut(Des) \subset Aut(C_{56,1})$. Since $|Aut(Des)| = 80640$ and $|Aut(C_{56,1})| = 161280$, by Lagrange's Theorem, it holds that $Aut(Des) \subset Aut(C_{56,1})$ and since $G \subset Aut(Des)$, then $Aut(Des) \sim= L_3(4).2^2$. Again, from the weight distribution tables, it holds that $C_{56,1}$ is the code $[56, 19, 18]_3$ which is a self-orthogonal code. □

Acknowledgement

The authors acknowledge the referees for their contribution in related lines of study.

Competing Interests

Authors have declared that no competing interests exist.

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