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Hyperbolic Jacobsthal Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, we introduce the Hyperbolic Jacobsthal numbers and we present recurrence relations, Binet's formulas, generating functions and the summation formulas for these numbers. Moreover, we investgate Lorentzian inner product for the hyperbolic Jacobsthal vectors.

Keywords: Jacobsthal numbers; hyperbolic numbers; hyperbolic jacobsthal numbers.

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1 Introduction and Preliminaries

In this paper, we define Hyperbolic Jacobsthal numbers in the next section and give some properties of them. First, in this section, we present some background about Hyperbolic numbers and Jacobsthal numbers. See for example, [1], [2], [3], [4], [5], [6], [7] and [8].

Jacobsthal sequence $\{J_n\}_{n\geq 0}$is defined by the second-order recurrence relation

$$J_n = J_{n-1} + 2J_{n-2} \tag{1.1}$$

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with the initial values $J_0 = 0, J_1 = 1$. Jacobsthal numbers are

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, ...$$

Binet's formula, generating functions, Simson formula, summation formula of Jacobsthal numbers are

$$\begin{aligned} J_n &= \frac{1}{3}\left(2^n - (-1)^n\right), \\ \sum_{n=1}^{\infty} J_n x^{n-1} &= \frac{1}{1 - x - 2x^2}, \\ J_{n+1}J_{n-1} - J_n^2 &= (-1)^n 2^{n-1}, \\ \sum_{k=0}^{n} J_k &= \frac{1}{2}(J_{n+2} - 1) \end{aligned}$$

respectively.

The set of hyperbolic numbers $\mathbb{H}$ can be described as

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R},\ h^2 = 1, x, y \in \mathbb{R}\}.$$

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$\begin{aligned} z_1 \pm z_2 &= (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2), \\ z_1 \times z_2 &= (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h(x_1y_2 + y_1x_2). \end{aligned}$$

and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2} = \frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{x_1y_2 + y_1x_2}{x_2^2 - y_2^2}.$$

The hyperbolic conjugation of $z = x + hy$ is defined by

$$\overline{z} = z^{\dagger} = x - hy.$$

Note that $\overline{\overline{z}} = z$. Note also that for any hyperbolic numbers z_1, z_2, z we have

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2}, \\ \|z\|^2 &= z \times \overline{z} = x^2 - y^2. \end{aligned}$$

For more information on hyperbolic numbers, see for example, [9], [10], [11], [12], [13], [14] and [15].

2 Hyperbolic Jacobsthal Sequence

In [16], author defined hyperbolic Fibonacci sequence and investigated its properties. In this work we define hyperbolic Jacobsthal sequence and investigated its properties. For dual Jacobsthal sequence see [17].

The hyperbolic Jacobsthal numbers are defined by

$$\widetilde{J}_n = J_n + hJ_{n+1} \tag{2.1}$$

with initial conditions $\widetilde{J}_0 = h$, $\widetilde{J}_1 = 1 + h$ where $h^2 = 1$. Then the hyperbolic Jacobsthal numbers are

$$h, 1 + h, 1 + 3h, 3 + 5h, 5 + 11h, 11 + 21h, 21 + 43h, 43 + 85h, ...$$

It can be easily shown that

$$\widetilde{J}_n = \widetilde{J}_{n-1} + 2\widetilde{J}_{n-2}. \tag{2.2}$$

To see this, we have

$$\begin{aligned} \widetilde{J}_n &= J_n + hJ_{n+1} = J_{n-1} + 2J_{n-2} + h(J_n + 2J_{n-1}) \\ &= (J_{n-1} + hJ_n) + 2(J_{n-2} + hJ_{n-1}) = \widetilde{J}_{n-1} + 2\widetilde{J}_{n-2}. \end{aligned}$$

Now, we start to give some results starting with the following Theorem. See [17].

Theorem 2.1. *If $\widetilde{J}_n$ is a hyperbolic Jacobsthal number, then*

$$\lim_{n\to\infty} \frac{\widetilde{J}_{n+1}}{\widetilde{J}_n} = 2.$$

Proof. For the Jacobsthal sequence J_n we have

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = 2.$$

Then using this limit value for the hyperbolic Jacobsthal numbers $\widetilde{J}_n$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{\widetilde{J}_{n+1}}{\widetilde{J}_n} &= \lim_{n\to\infty} \frac{J_{n+1} + hJ_{n+2}}{J_n + hJ_{n+1}} \\ &= \lim_{n\to\infty} \frac{J_{n+1} + h(J_{n+1} + 2J_n)}{J_n + hJ_{n+1}} \\ &= \lim_{n\to\infty} \frac{\frac{J_{n+1}}{J_n} + h(\frac{J_{n+1}}{J_n} + 2)}{1 + h\frac{J_{n+1}}{J_n}} \\ &= \frac{2 + h(2+2)}{1+2h} = \frac{2+4h}{1+2h} = 2. \end{aligned}$$

Next, we present Binet's formula. See [17].

Theorem 2.2. *The Binet's formula for the hyperbolic Jacobsthal sequence is given as*

$$\widetilde{J}_n = \frac{1}{3}\left((1+2h)2^n - (1-h)(-1)^n\right). \tag{2.3}$$

Proof. Using Binet's formula $J_n = \frac{1}{3}\left(2^n - (-1)^n\right)$ we see that

$$\begin{aligned} \widetilde{J}_n &= J_n + hJ_{n+1} \\ &= \frac{1}{3}\left(2^n - (-1)^n\right) + h\frac{1}{3}\left(2^{n+1} - (-1)^{n+1}\right) \\ &= \frac{1}{3}(-1)^n(h-1) + \frac{1}{3}2^n(1+2h) \\ &= \frac{1}{3}\left((1+2h)2^n - (1-h)(-1)^n\right). \end{aligned}$$

It is useful to let Binet's formula for the hyperbolic Jacobsthal sequence as follows.

Corollary 2.3. *Binet's formula for the hyperbolic Jacobsthal sequence can be given as*

$$\widetilde{J}_n = \frac{1}{3}(1+2h)\left(2^n + (1-h)(-1)^n\right). \tag{2.4}$$

Proof. Using Binet's formula for the hyperbolic Jacobsthal sequence (2.3), we see that

$$\begin{aligned}
\widetilde{J}_n &= \frac{1}{3}\left((1+2h)2^n - (1-h)(-1)^n\right) = \frac{1}{3}(1+2h)\left(2^n - \left(\frac{1-h}{1+2h}\right)(-1)^n\right) \\
&= \frac{1}{3}(1+2h)\left(2^n - \left(\frac{1-h}{1-4h^2}\right)(1-2h)(-1)^n\right) \\
&= \frac{1}{3}(1+2h)\left(2^n - \left(\frac{1-2h-h+2h^2}{-3}\right)(-1)^n\right) \\
&= \frac{1}{3}(1+2h)\left(2^n + (1-h)(-1)^n\right).
\end{aligned}$$

Next, we present the generating function for the hyperbolic Jacobsthal numbers. For the generating function of Jacobsthal numbers see [18], [19].

Theorem 2.4. *The generating function for the hyperbolic Jacobsthal numbers is*

$$\sum_{n=0}^{\infty} \widetilde{J}_n x^n = \frac{h+x}{1-x-2x^2}. \tag{2.5}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \widetilde{J}_n x^n$$

be generating function of hyperbolic Jacobsthal numbers. Then

$$\begin{aligned}
(1-x-2x^2)g(x) &= \sum_{n=0}^{\infty} \widetilde{J}_n x^n - x\sum_{n=0}^{\infty} \widetilde{J}_n x^n - 2x^2\sum_{n=0}^{\infty} \widetilde{J}_n x^n \\
&= \sum_{n=0}^{\infty} \widetilde{J}_n x^n - \sum_{n=0}^{\infty} \widetilde{J}_n x^{n+1} - 2\sum_{n=0}^{\infty} \widetilde{J}_n x^{n+2} \\
&= \sum_{n=0}^{\infty} \widetilde{J}_n x^n - \sum_{n=1}^{\infty} \widetilde{J}_{n-1} x^n - 2\sum_{n=2}^{\infty} \widetilde{J}_{n-2} x^n \\
&= (\widetilde{J}_0 + \widetilde{J}_1 x) - \widetilde{J}_0 x - \sum_{n=2}^{\infty} (\widetilde{J}_n - \widetilde{J}_{n-1} - 2\widetilde{J}_{n-2}) x^n \\
&= \widetilde{J}_0 + \widetilde{J}_1 x - \widetilde{J}_0 x \\
&= \widetilde{J}_0 + (\widetilde{J}_1 - \widetilde{J}_0) x.
\end{aligned}$$

Rearranging the above equation, we obtain

$$g(x) = \frac{\widetilde{J}_0 + (\widetilde{J}_1 - \widetilde{J}_0)x}{1-x-2x^2} = \frac{h+x}{1-x-2x^2}$$

since $\widetilde{J}_0 = h$, $\widetilde{J}_1 = 1+h$.

Next, we give linear sum identitity of hyperbolic Jacobsthal numbers.

Theorem 2.5. *For* $n \geq 0$, *we have the following formula:*

$$\sum_{k=0}^{n} \widetilde{J}_k = \frac{1}{2}(\widetilde{J}_{n+2} - (h+1)). \tag{2.6}$$

Proof. The proof follows from the summing formula

$$\sum_{k=0}^{n} J_k = \frac{1}{2}(J_{n+2} - 1).$$

We now present a few special identities for the hyperbolic Jacobsthal sequence $\{\widetilde{J}_n\}$.

Theorem 2.6. *(Catalan's identity) For all natural numbers n and m, the following identity holds*

$$\widetilde{J}_{n+m}\widetilde{J}_{n-m} - \widetilde{J}_n^2 = \frac{1}{9}(-2)^{n-m}((-1)^m - 2^m)^2(1-h).$$

Proof. We use the Binet's formula (2.3)

$$\widetilde{J}_n = \frac{1}{3}((1+2h)2^n - (1-h)(-1)^n).$$

Then we have

$$\begin{aligned}
\widetilde{J}_{n+m}\widetilde{J}_{n-m} - \widetilde{J}_n^2 &= \frac{1}{3}\left((1+2h)2^{n+m} - (1-h)(-1)^{n+m}\right)\frac{1}{3}\left((1+2h)2^{n-m} - (1-h)(-1)^{n-m}\right) \\
&\quad -\frac{1}{9}((1+2h)2^n - (1-h)(-1)^n)^2 \\
&= -\frac{1}{9}(-2)^{n-m}((-1)^m - 2^m)^2(1+2h)(1-h) \\
&= -\frac{1}{9}(-2)^{n-m}((-1)^m - 2^m)^2\left(-2h^2 + h + 1\right) \\
&= \frac{1}{9}(-2)^{n-m}((-1)^m - 2^m)^2(1-h).
\end{aligned}$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the hyperbolic Jacobsthal numbers.

Corollary 2.7. *(Cassini's identity) For all natural numbers n, the following identity holds*

$$\widetilde{J}_{n+1}\widetilde{J}_{n-1} - \widetilde{J}_n^2 = (-2)^{n-1}(1-h).$$

The d'Ocagne's, Gelin-Cesàro's and Melham's identities can also be obtained by using (2.3) or (2.4). The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of hyperbolic Jacobsthal sequence $\{\widetilde{J}_n\}$.

Theorem 2.8. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\widetilde{J}_{m+1}\widetilde{J}_n - \widetilde{J}_m\widetilde{J}_{n+1} = \frac{1}{3}(1-h)((-1)^n 2^m - (-1)^m 2^n),$$

(b) *(Gelin-Cesàro's identity)*

$$\widetilde{J}_{n+2}\widetilde{J}_{n+1}\widetilde{J}_{n-1}\widetilde{J}_{n-2} - \widetilde{J}_n^4 = \frac{1}{9}(-2)^n(h-1)\left(2^{n-2} + 13(-2)^{n-2} + 1\right),$$

(c) *(Melham's identity)*

$$\widetilde{J}_{n+1}\widetilde{J}_{n+2}\widetilde{J}_{n+6} - \widetilde{J}_{n+3}^3 = \frac{1}{3}(h-1)(-2)^{n+2}\left(2^{n+1} - 5(-1)^n\right).$$

Proof.

(a) Using $\widetilde{J}_n = \frac{1}{3}\left((1+2h)2^n - (1-h)(-1)^n\right)$ and saying $(1+2h) = a$ and $(1-h) = b$, we have

$$\begin{aligned}
\widetilde{J}_{m+1}\widetilde{J}_n - \widetilde{J}_m\widetilde{J}_{n+1} &= \frac{1}{9}\left[\left(a2^{m+1} - b(-1)^{m+1}\right)\left(a2^n - b(-1)^n\right) - \left(a2^m - b(-1)^m\right)\left(a2^{n+1} - b(-1)^{n+1}\right)\right] \\
&= \frac{1}{9}\left[a^2 2^{m+n+1} + b^2(-1)^{m+n+1} - ab\left((-1)^{m+1}2^n + 2^{m+1}(-1)^n\right)\right] \\
&\quad -\frac{1}{9}\left[a^2 2^{m+n+1} + b^2(-1)^{m+n+1} - ab\left((-1)^m 2^{n+1} + 2^m(-1)^{n+1}\right)\right] \\
&= \frac{1}{9}(1-h)\left[(-1)^{m+1}2^n + 2^{m+1}(-1)^n - (-1)^m 2^{n+1} - 2^m(-1)^{n+1}\right] \\
&= \frac{1}{9}(1-h)\left[3(-1)^n 2^m - 3(-1)^m 2^n\right] \\
&= \frac{1}{3}(1-h)\left[(-1)^n 2^m - (-1)^m 2^n\right].
\end{aligned}$$

(b) Using $\widetilde{J}_n = \frac{1}{3}(1+2h)\left(2^n + (1-h)(-1)^n\right)$ and saying $2^n = a$ and $(1-h)(-1)^n = b$, we have

$$\begin{aligned}
\widetilde{J}_{n+2}\widetilde{J}_{n+1}\widetilde{J}_{n-1}\widetilde{J}_{n-2} - \widetilde{J}_n^4 &= \frac{1}{81}(1+2h)^4\left[(4a+b)(2a-b)\left(\frac{a}{2} - b\right)\left(\frac{a}{4} + b\right) - (a+b)^4\right] \\
&= \frac{1}{81}(41+40h)\left[\left(8a^2 - 2ab - b^2\right)\left(\frac{a^2}{8} - \frac{ab}{2} - b^2\right)\right. \\
&\quad \left. - \left(a^4 + b^4 + 4a^3b + 6a^2b^2 + 4ab^3\right)\right] \\
&= \frac{1}{81}(41+40h)\left[\left(\frac{7}{4}a^3b + \frac{7}{4}ab^3 - \frac{69}{8}a^2b^2\right) - \left(4a^3b + 6a^2b^2 + 4ab^3\right)\right] \\
&= \frac{1}{81}(41+40h)\left[\frac{-9}{4}a^3b + \frac{-9}{4}ab^3 - \frac{117}{8}a^2b^2\right] \\
&= \frac{1}{9}(41+40h)(1-h)(-2)^n\left[\frac{-1}{4}\left(a^2 + b^2\right) - \frac{13}{8}ab\right] \\
&= \frac{1}{9}(1-h)(-2)^n\left[\frac{-1}{4}\left(2^{2n} + 2(1-h)\right) - \frac{13}{8}(-2)^n(1-h)\right] \\
&= \frac{1}{9}\left[\frac{-1}{4}\left((-2)^{3n} + 4(1-h)(-2)^n\right) - \frac{13}{4}(-2)^{2n}(1-h)\right] \\
&= \frac{1}{9}(-2)^n(h-1)\left[2^{2n-2} + 13(-2)^{n-2} + 1\right].
\end{aligned}$$

(c) Using $\widetilde{J}_n = \frac{1}{3}(1+2h)\left(2^n + (1-h)(-1)^n\right)$ and saying $2^n = a$ and $(1-h)(-1)^n = b$, we have

$$\begin{aligned}
\widetilde{J}_{n+1}\widetilde{J}_{n+2}\widetilde{J}_{n+6} - \widetilde{J}_{n+3}^3 &= \frac{1}{27}(1+2h)^3\left[(2a-b)(4a+b)(64a+b) - (8a-b)^3\right] \\
&= \frac{1}{27}(13+14h)\left[\left(8a^2 - 2ab - b^2\right)(64a+b) - \left(512a^3 - 192a^2b + 24ab^2 - b^3\right)\right] \\
&= \frac{1}{27}(13+14h)\left[\left(-120a^2b - 66ab^2\right) - \left(-192a^2b + 24ab^2\right)\right] \\
&= \frac{1}{27}(13+14h)\left[72a^2b - 90ab^2\right] \\
&= \frac{1}{27}(13+14h)(1-h)(-2)^n\left[72.2^n - 90(1-h)(-1)^n\right] \\
&= \frac{1}{3}(h-1)(-2)^n\left(8.2^n - 10(1-h)(-1)^n\right) \\
&= \frac{1}{3}(-2)^n\left(8.2^n(h-1) - 20(h-1)(-1)^n\right) \\
&= \frac{1}{3}(h-1)(-2)^{n+2}\left(2^{n+1} - 5(-1)^n\right).
\end{aligned}$$

3 Hyperbolic Jacobsthal Vectors

Suppose that $\overrightarrow{z_1} = (x_1, x_2, x_3)$ and $\overrightarrow{z_2} = (y_1, y_2, y_3)$ are vectors in $\mathbb{R}^3$. The Lorentzian inner product of z_1 and z_2 is defined as, see for example [16] and [20],

$$z_1 z_2 = \langle \overrightarrow{z_1}, \overrightarrow{z_2} \rangle_L = x_1y_1 + x_2y_2 - x_3y_3. \tag{3.1}$$

Note that for $\overrightarrow{z} = (x_1, x_2, x_3)$ we have

$$\langle \overrightarrow{z}, \overrightarrow{z} \rangle_L = x_1^2 + x_2^2 - x_3^2$$

and

$$\|\overrightarrow{z}\|^2 = x_1^2 + x_2^2 - x_3^2.$$

The vector space $\mathbb{R}^3$ together with the Lorentzian inner product $\langle .,, \rangle_L$ is called Lorentzian inner product space and usually denoted by $\mathbb{L}^{2,1}$ or $\mathbb{L}^3$.

A hyperbolic Jacobsthal vector is defined by

$$\overrightarrow{\widetilde{J}}_n = \left(\widetilde{J}_n, \widetilde{J}_{n+1}, \widetilde{J}_{n+2}\right).$$

From equations (2.1) and (2.2) we see that

$$\overrightarrow{\widetilde{J}}_n = \overrightarrow{J}_n + h\overrightarrow{J}_{n+1}$$

where $\overrightarrow{J}_n = (J_n, J_{n+1}, J_{n+2})$ and $\overrightarrow{J}_{n+1} = (J_{n+1}, J_{n+2}, J_{n+3})$ are Jacobsthal vectors and $h^2 = 1$.

The product of the hyperbolic Jacobsthal vector $\overrightarrow{\widetilde{J}}_n$ and the scalar $\lambda \in \mathbb{R}$ is given by

$$\lambda \overrightarrow{\widetilde{J}}_n = \lambda \overrightarrow{J}_n + h\lambda \overrightarrow{J}_{n+1}$$

and $\overrightarrow{\widetilde{J}}_n$ and $\overrightarrow{\widetilde{J}}_m$ are equal if and only if

$$\begin{aligned} J_n &= J_m \\ J_{n+1} &= J_{m+1} \\ J_{n+2} &= J_{m+2}. \end{aligned}$$

Note that

$$\begin{aligned} \overrightarrow{\widetilde{J}}_0 &= \left(\widetilde{J}_0, \widetilde{J}_1, \widetilde{J}_2\right) = (J_0 + hJ_1, J_1 + hJ_2, J_2 + hJ_3) = (h, 1+h, 1+3h), \\ \overrightarrow{\widetilde{J}}_1 &= \left(\widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3\right) = (J_1 + hJ_2, J_2 + hJ_3, J_3 + hJ_4) = (1+h, 1+3h, 3+5h), \\ \overrightarrow{\widetilde{J}}_2 &= \left(\widetilde{J}_2, \widetilde{J}_3, \widetilde{J}_4\right) = (J_2 + hJ_3, J_3 + hJ_4, J_4 + hJ_5) = (1+3h, 3+5h, 5+11h). \end{aligned}$$

Now, we give Lorentzian inner product of hyperbolic Jacobsthal vectors.

Theorem 3.1. *Let $\overrightarrow{\widetilde{J}}_n$ and $\overrightarrow{\widetilde{J}}_m$ be two hyperbolic Jacobsthal vectors. The Lorentzian inner product of $\overrightarrow{\widetilde{J}}_n$ and $\overrightarrow{\widetilde{J}}_m$ is given by*

$$\begin{aligned} \left\langle \overrightarrow{\widetilde{J}}_n, \overrightarrow{\widetilde{J}}_m \right\rangle_L &= (J_m J_n + 2J_{m+1}J_{n+1} - J_{m+3}J_{n+3}) \qquad (3.2) \\ &\quad + h(J_{m+1}J_{n+2} + J_{m+2}J_{n+1} - J_{m+2}J_{n+3} - J_{m+3}J_{n+2} + J_n J_{m+1} + J_m J_{n+1}). \end{aligned}$$

Proof. The Lorentzian inner product of $\overrightarrow{\widetilde{J}}_n = \left(\widetilde{J}_n, \widetilde{J}_{n+1}, \widetilde{J}_{n+2}\right)$ and $\overrightarrow{\widetilde{J}}_m = \left(\widetilde{J}_m, \widetilde{J}_{m+1}, \widetilde{J}_{m+2}\right)$ is defined by

$$\begin{aligned} \left\langle \overrightarrow{\widetilde{J}}_n, \overrightarrow{\widetilde{J}}_m \right\rangle_L &= \widetilde{J}_n \widetilde{J}_m + \widetilde{J}_{n+1}\widetilde{J}_{m+1} - \widetilde{J}_{n+2}\widetilde{J}_{m+2} \\ &= \left\langle \overrightarrow{J}_n, \overrightarrow{J}_m \right\rangle + \left\langle \overrightarrow{J}_{n+1}, \overrightarrow{J}_{m+1} \right\rangle + h\left[\left\langle \overrightarrow{J}_n, \overrightarrow{J}_{m+1} \right\rangle + \left\langle \overrightarrow{J}_{n+1}, \overrightarrow{J}_m \right\rangle\right] \end{aligned}$$

where $\overrightarrow{J}_n = (J_n, J_{n+1}, J_{n+2})$ is Jacobsthal vector. Using (3.1), we obtain

$$\begin{aligned} \left\langle \overrightarrow{J}_n, \overrightarrow{J}_m \right\rangle &= J_n J_m + J_{n+1} J_{m+1} - J_{n+2} J_{m+2}, \\ \left\langle \overrightarrow{J}_{n+1}, \overrightarrow{J}_{m+1} \right\rangle &= J_{n+1} J_{m+1} + J_{n+2} J_{m+2} - J_{n+3} J_{m+3}, \\ \left\langle \overrightarrow{J}_n, \overrightarrow{J}_{m+1} \right\rangle &= J_n J_{m+1} + J_{n+1} J_{m+2} - J_{n+2} J_{m+3}, \\ \left\langle \overrightarrow{J}_{n+1}, \overrightarrow{J}_m \right\rangle &= J_{n+1} J_m + J_{n+2} J_{m+1} - J_{n+3} J_{m+2}. \end{aligned}$$

Using the last four equations, we have the required equation (3.2).

4 Conclusion

The hyperbolic Jacobsthal numbers are defined by

$$\widetilde{J}_n = J_n + hJ_{n+1} \tag{4.1}$$

with initial conditions $\widetilde{J}_0 = h$, $\widetilde{J}_1 = 1 + h$ where $h^2 = 1$.

We introduced the hyperbolic Jacobsthal numbers and we presented recurrence relations, Binet's formulas, generating functions and the summation formulas for these numbers. Moreover, we investgated the Lorentzian inner product for the hyperbolic Jacobsthal vectors.

There are new studies on dual hyperbolic Fibonacci and Lucas numbers (see [21]), and on dual hyperbolic generalized Fibonacci numbers (see [22]) by other authors.

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Competing Interests

Author has declared that no competing interests exist.

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