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A Study on Properties and Applications of a Lomax Gompertz-Makeham Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The article presents an extension of the Gompertz-Makeham distribution using the Lomax generator of probability distributions. This generalization of the Gompertz-Makeham distribution provides a more skewed and flexible compound model called Lomax Gompertz-Makeham distribution. The paper derives and discusses some Mathematical and Statistical properties of the new distribution. The unknown parameters of the new model are estimated via the method of maximum likelihood estimation. In conclusion, the new distribution is applied to two real life datasets together with two other related models to check its flexibility or performance and the results indicate that the proposed extension is more flexible compared to the other two distributions considered in the paper based on the two datasets used.

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1 Introduction

The Gompertz-Makeham distribution (GOMAD) was introduced by Makeham in 1860 [1]. It is an extension of the Gompertz probability distribution that was introduced by Gompertz in 1825 [2]. The GOMAD is a continuous probability distribution that has been widely used in survival analysis, modeling human mortality, constructing actuarial tables and growth models. It has been recently used in many fields of sciences including actuaries, biology, demography, gerontology, and computer science.

A comprehensive review of the history and theory of the GOMAD can be found in Marshall and Olkin [3]. Golubev [4] emphasizes the practical importance of this probability distribution. Detailed information about the GOMAD, its mathematical and statistical properties, and its applications can be found in Johnson et al. [5] and Dey et al. [6].

There exist a good number of families of probability distributions that are used for adding parameters to existing distributions while making them more flexible for modeling heavily skewed dataset. A summary of these families include: The beta generated family (Beta-G) by Eugene et al. [7], Transmuted family of distributions by Shaw and Buckley [8], Gamma-G (type 1) by Zografos and Balakrishnan [9], the Kumaraswamy-G by Cordeiro and de Castro [10], McDonald-G by Alexander et al. [11], Gamma-G (type 2) by Ristic et al. [12], Gamma-G (type 3) by Torabi and Montazari [13], Log-gamma-G by Amini et al. [14], Exponentiated T-X by Alzaghal et al. [15], Exponentiated-G (EG) by Hassan and Elgarhy [16], Weibull-X by Alzaatreh et al. [17], Weibull-G by Bourguignon et al. [18], Logistic-G by Torabi and Montazari [19], Gamma-X by Alzaatreh et al. [20], a Lomax-G family by Cordeiro et al. [21], a new generalized Weibull-G family by Cordeiro et al. [22], Beta Marshall-Olkin family of distributions by Alizadeh et al. [23], Logistic-X by Tahir et al. [24], a new Weibull-G family by Tahir et al. [25], a Lindley-G family by Cakmakyapan and Ozel [26], a Gompertz-G family by Alizadeh et al. [27] and Odd Lindley-G family by Gomes-Silva et al. [28] and so on.

Sequel to the introduction of the families of probability distribution above and the will to add skewness and flexibility to classical distributions particularly the Gompertz-Makeham distribution, many authors have proposed different extensions of this distribution and some of the recent and known studies include the the Kumaraswamy Gompertz-Makeham distribution by Chukwu and Ogunde [29], the transmuted Gompertz-Makeham distribution by El-Bar [30] using the quadratic rank transmutation map by Shaw and Buckley [8], the Cubic Transmuted Gompertz-Makeham Distribution by Riffi and Hamdan [31] and the generalized transmuted Gompertz-Makeham distribution by Riffi [32].

Research has revealed that using Lomax generator of probability distributions (Lomax-G family) by Cordeiro et al. [21] to add two parameters to a continuous distribution produces a compound distribution with greater skewness and flexibility for modeling real life datasets (Venegas et al. [33], Omale et al. [34], Ieren et al. [35], Ieren and Kuhe [36]).

Therefore, motivated by the above information, the main interest in this article is to present another generalization of the Gompertz-Makeham distribution using the Lomax generator of probability distributions proposed by Cordeiro et al. [21] and which has been used recently by Venegas et al. [33], Omale et al. [34], Ieren and Kuhe [36] and Ieren et al. [35]. It is the hope of the researchers that it will give a better compound model for analyzing real life situations especially in survival analysis, human mortality modeling, constructing actuarial tables and growth models.

The cumulative distribution function (c.d.f) and probability density function (pdf) of the Gompertz-Makeham distribution are defined as:

$$G(x)=1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \tag{1}$$

and

$$g(x)=\left[\theta+\alpha e^{\beta x}\right]e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \tag{2}$$

respectively, for $x>0,\alpha,\beta,\theta>0,$ where θ is the scale parameter and α and β are the shape parameters of Gompertz-Makeham distribution.

This article is organized in sections as follows: definition of the new distribution with its validity and graphical analysis is provided in section 2. Section 3 derived some Mathematical and Statistical properties of the new distribution. The estimation of unknown parameters of the distribution using maximum likelihood estimation is provided in section 4. An application of the new model to two real life datasets is done in section 5 and a very useful summary as well as conclusion is offered in section 6.

2 Formulation of the Lomax Gompertz-Makeham Distribution (LOGOMAD)

2.1 Definition

According to Cordeiro et al. [21], the cumulative distribution function of the proposed Lomax generator of distributions (also known as "Lomax-G family of distributions) is defined as

$$F(x)=\int_{0}^{-\log[1-G(x)]} ab^{a}\frac{dt}{(b+t)^{a+1}} \tag{3}$$

where $G(x)$ is the cdf of any continuous distribution to be modified or generalized and $a>0$ and $b>0$ are the two extra shape parameters of the Lomax-G family.

Using integration by substitution in equation (3) above and evaluating the integrand in the equation yields

$$F(x)=1-\left\{\frac{b}{b-\log\left[1-G(x)\right]}\right\}^{a} \tag{4}$$

The corresponding probability density function (pdf) of the Lomax-G family is obtained from equation (4) by taking the derivative of the cdf, $F(x)$ with respect to *x* and is given as:

$$f(x)=ab^{a}\frac{g(x)}{\left[1-G(x)\right]\left\{b-\log\left[1-G(x)\right]\right\}^{a+1}} \tag{5}$$

where $g(x)$ and $G(x)$ represent the *pdf* and the *cdf* of the continuous distribution to be modified, extended or generalized respectively. Also note that the major benefit of (5) is to offer more flexibility and skewness to the extremes of the *pdf* and therefore makes it more suitable for analyzing data with high degree of asymmetry.

Substituting equation (1) and (2) in (4) and (5) above and simplifying, we obtain the *cdf* and *pdf* of the LOGOMAD for a random variable *X* as:

$$F(x) = 1 - \left\{ \frac{b}{b - \log\left[1 - \left(1 - \mathrm{e}^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]} \right\}^{a} = 1 - b^{a}\left\{ b - \log\left[1 - \left(1 - \mathrm{e}^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\right\}^{a} \tag{6}$$

and

$$f(x) = ab^{a}\frac{\left[\theta + \alpha e^{\beta x}\right]e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}}{\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\left\{b - \log\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\right\}^{a+1}} \tag{7}$$

respectively. For $x > 0; a, b, \alpha, \beta, \theta > 0$.

2.2 Validity of the model *f(x)*

Recall that for any valid continuous probability distribution, the following integral in (8) must holds, that is

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{8}$$

PROOF

Considering the *pdf* of the LOGOMAD, which is given as

$$f(x) = ab^{a}\frac{\left[\theta + \alpha e^{\beta x}\right]e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}}{\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\left\{b - \log\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\right\}^{a+1}}$$

Substituting this pdf in equation (8) above and simplifying, we have

$$\int_{0}^{\infty} f(x)dx = \int_{0}^{\infty}\left(ab^{a}\frac{\left[\theta + \alpha e^{\beta x}\right]e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}}{\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\left\{b - \log\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\right\}^{a+1}}\right)dx$$

$$\int_0^\infty f(x)dx = \int_0^\infty \left(\frac{ab^a\left[\theta+\alpha e^{\beta x}\right]e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}b^{-(a+1)}\left\{1-b^{-1}\log\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\right\}^{-(a+1)}}{\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]} \right)dx \tag{9}$$

Now, from equation (9), let

$$y = 1-b^{-1}\log\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]$$

Such that

$$\frac{dy}{dx} = \frac{\left[\theta+\alpha e^{\beta x}\right]e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{b\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]}$$

Which implies that

$$dx = \frac{b\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]dy}{\left[\theta+\alpha e^{\beta x}\right]e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}$$

Substituting for dx in (9) and simplifying the resulting expression, we obtain

$$\int_0^\infty f(x)dx = \int_0^\infty ay^{-(a+1)}dy \tag{10}$$

Integrating and applying the limit in equation (10) above results in the following:

$$\int_0^\infty f(x)dx = \int_0^\infty ay^{-(a+1)}dy = -\left[y^{-a}\right]_0^\infty \tag{11}$$

But recall that

$$y = 1-b^{-1}\log\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]$$

Hence, substituting for y in equation (11) and simplifying will result in the following:

$$\int_0^\infty f(x)dx = -\left[y^{-a}\right]_0^\infty = -\left\{\lim_{x\to\infty}\left(1-b^{-1}\log\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\right)^{-a} - \lim_{x\to 0}\left(1-b^{-1}\log\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\right)^{-a}\right\} \tag{12}$$

Recall that $1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ is the cdf of the Gompertz-Makeham distribution and its limit as X approaches infinity, $x\to\infty$ is equal to one (1) while its limit as X tends to zero, $x\to 0$ is equal to zero (0). Therefore, from equation (12), we have:

$$\int_0^\infty f(x)dx = -\left[y^{-a}\right]_0^\infty = -\left\{\left(1-b^{-1}\log[1-1]\right)^{-a}-\left(1-b^{-1}\log[1-0]\right)^{-a}\right\}$$

$$\int_0^\infty f(x)dx = -\left[y^{-a}\right]_0^\infty = -\left\{\left(1-b^{-1}\log[0]\right)^{-a}-\left(1-b^{-1}\log[1]\right)^{-a}\right\}$$

$$\int_0^\infty f(x)dx = -\left[y^{-a}\right]_0^\infty = -\left\{0-1\right\} = 1$$

Therefore,

$$\int_0^\infty f(x)dx = 1$$

and hence it is that the proposed pdf of the LOGOMAD in equation (7) is a valid probability density function.

2.3 Graphical presentation of Pdf and Cdf of LOGOMAD

The *pdf* and *cdf* of the LOGOMAD using some parameter values are displayed in **figure 1** and figure **2** respectively as follows.

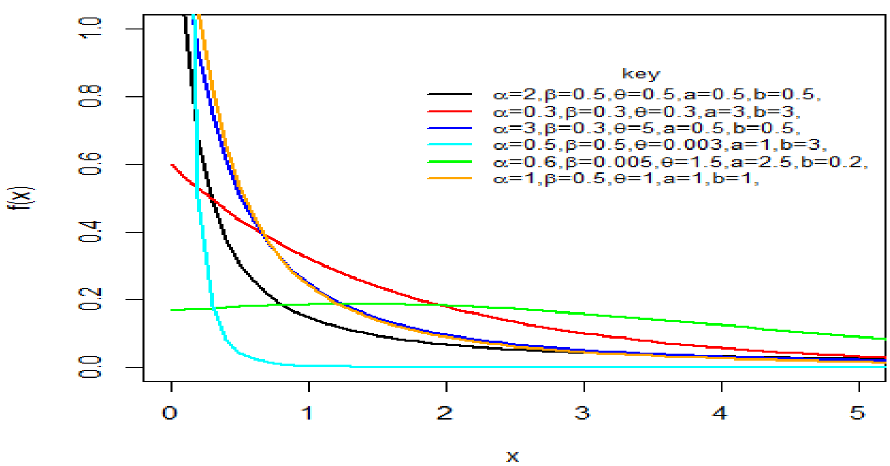

Fig. 1. ***PDF*** **of LOGOMAD**

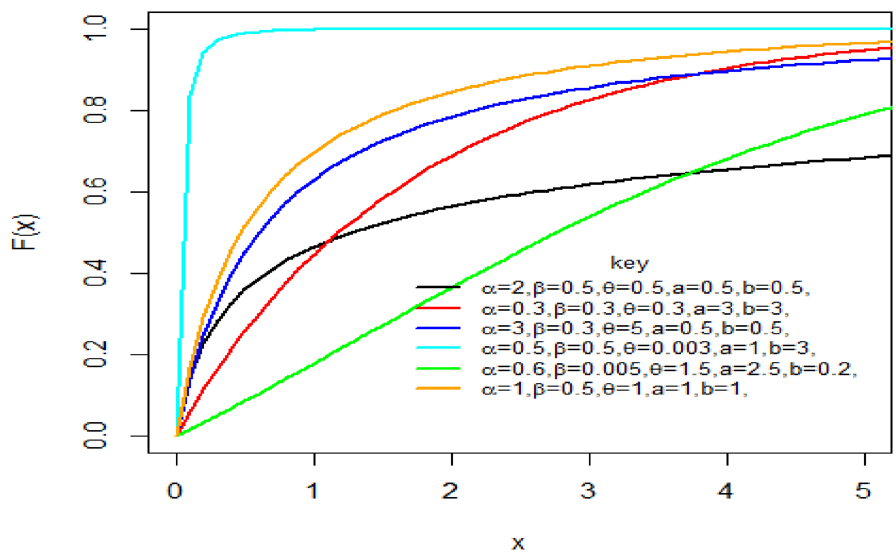

Fig. 2. ***CDF*** **of LOGOMAD**

3 Mathematical and Statistical Properties of LOGOMAD

This section contains derivations and discussions of some properties of the proposed distribution. These are presented as follows:

3.1 Asymptotic behavior

Here, the asymptotic properties of the LOGOMAD are being investigated, that is, the limit of the pdf and *cdf* of the LOGOMAD as X approaches infinity, $x \to \infty$ and as X tends to zero, $x \to 0$. This is done as follows:

LEMMA 1: The limit of the pdf, f(x) of the LOGOMAD as X approaches infinity, $x \to \infty$ is equal to zero (0) and the limit of the pdf, f(x) of the LOGOMAD as X tends to zero (0), $x \to 0$ is also equal to zero (0).

PROOF

(i) The limit of the pdf, f(x) of the LOGOMAD as X approaches infinity, $x \to \infty$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left\{ ab^a \frac{\left[\theta + \alpha e^{\beta x}\right] e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{\left[1-\left(1-e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\left\{b - \log\left[1-\left(1-e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\right\}^{a+1}} \right\} \tag{13}$$

Recall that $1-e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ is the cdf of the Gompertz-Makeham distribution and its limit as X approaches infinity, $x \to \infty$ is equal to one (1) and that $\left[\theta + \alpha e^{\beta x}\right] e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ is the pdf of the Gompertz-

Makeham distribution and its limit as X approaches infinity, $x \to \infty$ is equal to zero (0), therefore simplifying equation (13) above gives:

$$\lim_{x\to\infty} f(x) = ab^a \frac{(0)}{(1-1)\{b-\log(0)\}^{a+1}} = (0)\frac{ab^a}{(0)\{b-\log(0)\}^{a+1}} = 0 \tag{14}$$

(ii) The limit of the pdf, f(x) of the LOGOMAD as X tends to zero (0), $x \to 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left\{ ab^a \frac{\left[\theta+\alpha e^{\beta x}\right] e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\left\{b-\log\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\right\}^{a+1}}\right\} \tag{15}$$

Recall that $1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ is the cdf of the Gompertz-Makeham distribution and its limit as X tends to zero (0), $x \to 0$ is equal to zero (0) and also that $\left[\theta+\alpha e^{\beta x}\right] e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ is the pdf of the Gompertz-Makeham distribution and its limit as X tends to zero (0), $x \to 0$ is equal to zero (0), therefore simplifying the equation (15) above gives:

$$\lim_{x\to 0} f(x) = ab^a \frac{(0)}{(1)\{b-(0)\}^{a+1}} = ab^a \frac{(0)}{\{b\}^{a+1}} = a\frac{(0)}{b} = 0 \tag{16}$$

LEMMA 2: The limit of the cdf, F(x) of the LOGOMAD as X approaches infinity, $x \to \infty$ is equal to one (1) and limit of the cdf, F(x) of the LOGOMAD as X tends to zero (0), $x \to 0$ is equal to zero (0).

PROOF

(i) The limit of the cdf, F(x) of LOGOMAD as X approaches infinity, $x \to \infty$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left\{1-\left\{\frac{b}{b-\log\left[1-\left(1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]}\right\}^a\right\} \tag{17}$$

Recall that $1-e^{-\theta x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ is the cdf of the Gompertz-Makeham distribution and its limit as X approaches infinity, $x \to \infty$ is equal to one (1), therefore simplifying equation (17) above gives:

$$\lim_{x\to\infty} F(x) = 1-\left\{\frac{b}{b-\log[1-1]}\right\}^a = 1-\left\{\frac{b}{b-\log(0)}\right\}^a = 1 \tag{18}$$

(ii) The limit of the cdf, F(x) of LOGOMAD as X tends to zero (0), $x \to 0$

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \left\{ 1 - \left\{ \frac{b}{b - \log\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]} \right\}^{a} \right\} \tag{19}$$

Recall that $1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ is the cdf of the Gompertz-Makeham distribution and its limit as X tends to zero (0), $x \to 0$ is equal to zero (0), therefore simplifying equation (19) above gives:

$$\lim_{x\to 0} F(x) = 1 - \left\{ \frac{b}{b - \log\left[1 - (0)\right]} \right\}^{a} = 1 - \left\{ \frac{b}{b - \log(1)} \right\}^{a} = 1 - \left\{ \frac{b}{b} \right\}^{a} = 1 - 1 = 0 \tag{20}$$

The lemma above and its proof show that the LOGOMAD has at least a mode and that it is a valid probability distribution.

3.2 Moments

Let X denote a continuous random variable, the n^{th} ordinary moment or moment about the origin of X is given by:

$$\mu_n^{'} = E\left(X^n\right) = \int_0^{\infty} x^n f(x)dx \tag{21}$$

where f(x) is considered as the pdf of LOGOMAD and is previously defined in equation (7) as:

$$f(x) = ab^a \frac{\left[\theta + \alpha e^{\beta x}\right] e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\left\{b - \log\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right]\right\}^{a+1}}$$

Before substitution in (21), the pdf of LOGOMAD is being simplified as follows:

$$f(x) = ab^a \frac{\left[\theta + \alpha e^{\beta x}\right] e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\left\{b - \log\left[e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]\right\}^{a+1}}$$

$$f(x) = ab^a \left(\theta + \alpha e^{\beta x}\right)\left(b + \theta x + \tfrac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right)^{-(a+1)}$$

$$f(x) = \frac{a}{b}\left(\theta + \alpha e^{\beta x}\right)\left(1 + \tfrac{\theta}{b} x + \tfrac{\alpha}{b\beta}\left(e^{\beta x} - 1\right)\right)^{-(a+1)} \tag{22}$$

Let

$$A=\left(1+\tfrac{\theta}{b}x+\tfrac{\alpha}{b\beta}\left(e^{\beta x}-1\right)\right)^{-(a+1)} \tag{23}$$

Using the generalized binomial theorem on A gives:

$$\left(1+\tfrac{\theta}{b}x+\tfrac{\alpha}{b\beta}\left(e^{\beta x}-1\right)\right)^{-(a+1)}=\sum_{k=0}^{\infty}\binom{-a-1}{k}\left[\tfrac{\theta}{b}x+\tfrac{\alpha}{b\beta}\left(e^{\beta x}-1\right)\right]^{k} \tag{24}$$

Making use of the result in (24) above, equation (22) becomes:

$$f(x)=\frac{a}{b}\sum_{k=0}^{\infty}\binom{-a-1}{k}\left(\theta+\alpha e^{\beta x}\right)\left[\tfrac{\theta}{b}x+\tfrac{\alpha}{b\beta}\left(e^{\beta x}-1\right)\right]^{k} \tag{25}$$

Also, using the generalized binomial theorem, we can write the last term from the above result in equation (25) as:

$$\left[\tfrac{\theta}{b}x+\tfrac{\alpha}{b\beta}\left(e^{\beta x}-1\right)\right]^{k}=\sum_{l=0}^{k}\binom{k}{l}\left(\frac{\theta}{b}\right)^{k}\left(\frac{\alpha}{b\beta}\right)^{l}x^{k}\left(e^{\beta x}-1\right)^{l} \tag{26}$$

Making use of the result in (26) above in equation (25) and simplifying the result, we obtain:

$$f(x)=\frac{a}{b}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{-a-1}{k}\binom{k}{l}\left(\frac{\theta}{b}\right)^{k}\left(\frac{\alpha}{b\beta}\right)^{l}(-1)^{l}x^{k}\left(\theta+\alpha e^{\beta x}\right)\left(1-e^{\beta x}\right)^{l} \tag{27}$$

Again making use of the generalized binomial expansion on the last term from equation (27) above, we have:

$$\left(1-e^{\beta x}\right)^{l}=\sum_{m=0}^{l}(-1)^{m}\binom{l}{m}e^{m\beta x} \tag{28}$$

Hence, the pdf in equation (27) can again be written in its simple form as follows:

$$f(x)=\frac{a}{b}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{-a-1}{k}\binom{k}{l}\binom{l}{m}\left(\frac{\theta}{b}\right)^{k}\left(\frac{\alpha}{b\beta}\right)^{l}(-1)^{l+m}x^{k}\left(\theta+\alpha e^{\beta x}\right)e^{m\beta x}$$

$$f(x)=\gamma_{k,l,m}\left(\theta+\alpha e^{\beta x}\right)x^{k}e^{m\beta x} \tag{29}$$

where

$$\gamma_{k,l,m}=\frac{a}{b}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{-a-1}{k}\binom{k}{l}\binom{l}{m}\left(\frac{\theta}{b}\right)^{k}\left(\frac{\alpha}{b\beta}\right)^{l}(-1)^{l+m}$$

Now, using the simplified form of the pdf of the LOGOMAD in equation (29), the n[th] moment ordinary moment of the LOGOMAD is derived as follows:

$$\mu_n^{'} = E\left(X^n\right) = \int_0^\infty x^n f(x)dx = \int_0^\infty x^n \left(\gamma_{k,l,m}\left(\theta + \alpha \mathrm{e}^{\beta x}\right)x^k \mathrm{e}^{m\beta x}\right)dx$$

$$\mu_n^{'} = E\left(X^n\right) = \gamma_{k,l,m}\left[\theta\int_0^\infty x^{n+k}\mathrm{e}^{m\beta x}dx + \alpha\int_0^\infty x^{n+k}\mathrm{e}^{\beta(m+1)x}dx\right] \tag{30}$$

Using integration by substitution method in equation (30), we have:

Let $-u_1 = m\beta x \Rightarrow x = -\frac{u_1}{m\beta}; -\frac{du_1}{dx} = m\beta \Rightarrow dx = -\frac{du_1}{m\beta}$

$$-u_2 = \beta\left(m+1\right)x \Rightarrow x = -\frac{u_2}{\beta\left(m+1\right)}; -\frac{du_2}{dx} = \beta\left(m+1\right) \Rightarrow dx = -\frac{du_2}{\beta\left(m+1\right)}$$

Substituting for x, u and dx in equation (30) and simplifying; we have:

$$\mu_n^{'} = E\left(X^n\right) = \gamma_{k,l,m}\left[\left(-\frac{\theta}{m\beta}\right)\int_0^\infty x^{n+k}\mathrm{e}^{-u_1}du_1 + \left(-\frac{\alpha}{\beta\left(m+1\right)}\right)\int_0^\infty x^{n+k}\mathrm{e}^{-u_2}du_2\right]$$

$$\mu_n^{'} = E\left(X^n\right) = \gamma_{k,l,m}\left[\left(-\frac{1}{m\beta}\right)^{n+k+1}\theta\int_0^\infty u_1^{n+k}\mathrm{e}^{-u_1}du_1 + \left(-\frac{1}{\beta\left(m+1\right)}\right)^{n+k+1}\alpha\int_0^\infty u_2^{n+k}\mathrm{e}^{-u_2}du_2\right]$$

Recall that $\int_0^\infty t^{n-1}e^{-t}dt = \Gamma\left(n\right)$ and that $\int_0^\infty t^n e^{-t}dt = \int_0^\infty t^{n+1-1}e^{-t}dt = \Gamma\left(n+1\right)$

Hence, we obtain the n^{th} ordinary moment of X for the LOGOMAD as:

$$\mu_n^{'} = E\left(X^n\right) = \frac{\gamma_{k,l,m}}{\left(-1\right)^{-\left(n+k+1\right)}}\left[\frac{\theta\Gamma\left(n+k+1\right)}{\left(m\beta\right)^{n+k+1}} + \frac{\alpha\Gamma\left(n+k+1\right)}{\left(\beta\left(m+1\right)\right)^{n+k+1}}\right] \tag{31}$$

The Central Moments: The n^{th} central moment or moment about the mean of X, say μ_n, can be obtained as:

$$\mu_n = E\left(X - \mu_1^{'}\right)^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\mu_1^{'i}\mu_{n-i}^{'} \tag{32}$$

The variance of X is obtained from the central moment when $n=2$, that is, variance is central moment of order two ($n=2$).

$$Var(X)=\sigma^2=E\left(X-\mu_1^{'}\right)^2=E\left(X^2\right)-\left\{E\left(X\right)\right\}^2=\mu_2^{'}-\left\{\mu_1^{'}\right\}^2 \tag{33}$$

The mean ($\mu_1^{'}$), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) can be calculated from the ordinary and non-central moments using some well-known relationships as given below:

$$\mu_1^{'}=E\left(X\right)=\frac{\gamma_{k,l,m}}{\left(-1\right)^{-(k+2)}}\left[\frac{\theta\Gamma\left(k+2\right)}{\left(m\beta\right)^{k+2}}+\frac{\alpha\Gamma\left(k+2\right)}{\left(\beta\left(m+1\right)\right)^{k+2}}\right] \tag{34}$$

$$Var(X)=\sigma^2=\mu_2^{'}-\left\{\mu_1^{'}\right\}^2 \tag{35}$$

$$CV=\left\{\frac{\sigma^2}{\left(\mu_1^{'}\right)^2}\right\}^{\frac{1}{2}} \tag{36}$$

$$CS=E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^3=\frac{\mu_3}{(\sigma)^3} \tag{37}$$

$$CK=E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^4=\frac{\mu_4}{(\sigma)^4} \tag{38}$$

Moment Generating Function: The moment generating function of a random variable X can be obtained as

$$M_x(t)=E\left[e^{tx}\right]=\int_{-\infty}^{\infty}e^{tx}f(x)dx \tag{39}$$

Using power series expansion, we have:

$$e^{tx}=\sum_{r=0}^{\infty}\frac{(tx)^r}{r!}=\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r \tag{40}$$

Using the result above and simplifying the integral in (39), therefore we have;

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty}x^rf(x)dx=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right)M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty}x^rf(x)dx=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\mu_r^{'}\right]$$

Hence, the moment generating function of the LOGOMAD can also be expressed as:

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\frac{\gamma_{k,l,m}}{(-1)^{-(n+k+1)}}\left[\frac{\theta\Gamma(n+k+1)}{(m\beta)^{n+k+1}}+\frac{\alpha\Gamma(n+k+1)}{(\beta(m+1))^{n+k+1}}\right]\right] \quad (41)$$

Characteristics Function: The characteristics function is very useful and has important properties that are beneficial in statistical theory. It is particularly useful in analysis of linear combination of independent random variables.

A representation for the characteristics function is given by

$$\varphi_x(t)=E\left[e^{itx}\right]=E\left[\cos(tx)+i\sin(tx)\right]=E\left[\cos(tx)\right]+E\left[i\sin(tx)\right] \quad (42)$$

Recall from power series expansion that:

$$\cos(tx)=\sum_{r=0}^{\infty}\frac{(-1)^r t^{2r}}{(2r)!}x^{2r} \quad \text{and} \quad E\left[\cos(tx)\right]=\sum_{r=0}^{\infty}\frac{(-1)^r t^{2r}}{(2r)!}\mu'_{2r}$$

And also that:

$$\sin(tx)=\sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}}{(2r+1)!}x^{2r+1} \quad \text{and} \quad E\left[\sin(tx)\right]=\sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}}{(2r+1)!}\mu'_{2r+1}$$

Hence, substituting the results above gives:

$$\phi_x(t)=\sum_{r=0}^{\infty}\frac{(-1)^r t^{2r}}{(2r)!}\mu'_{2r}+i\sum_{n=0}^{\infty}\frac{(-1)^r t^{2r+1}}{(2r+1)!}\mu'_{2r+1} \quad (43)$$

where μ'_{2r} and μ'_{2r+1} are obtained as ordinary moments of X for $n=2r$ and $n=2r+1$ respectively and can be computed from μ'_n in equation (31).

3.3 Reliability analysis of the LOGOMAD

The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x)=1-F(x) \quad (44)$$

Applying the *cdf* of the LOGOMAD in (44) and simplifying the result, the survival function for the LOGOMAD is obtained as:

$$S(x) = 1 - \left\{ 1 - b^a \left(b + \theta x + \tfrac{\alpha}{\beta} \left(e^{\beta x} - 1 \right) \right)^{-a} \right\}$$

$$S(x) = b^a \left(b + \theta x + \tfrac{\alpha}{\beta} \left(e^{\beta x} - 1 \right) \right)^{-a} \tag{45}$$

The figure below is a plot for the survival function of the LOGOMAD using different parameter values.

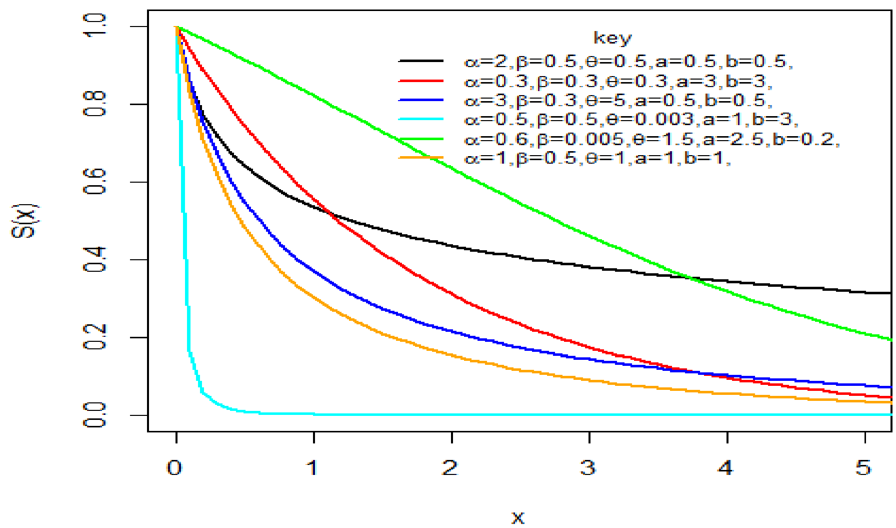

Fig. 3. Survival function of LOGOMAD

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{46}$$

Meanwhile, the expression for the hazard rate of the LOGOMAD is simplified and given by:

$$h(x) = \frac{ab^a \left(\theta + \alpha e^{\beta x} \right) \left(b + \theta x + \tfrac{\alpha}{\beta} \left(e^{\beta x} - 1 \right) \right)^{-(a+1)}}{b^a \left(b + \theta x + \tfrac{\alpha}{\beta} \left(e^{\beta x} - 1 \right) \right)^{-a}}$$

$$h(x) = \frac{a \left(\theta + \alpha e^{\beta x} \right)}{\left(b + \theta x + \tfrac{\alpha}{\beta} \left(e^{\beta x} - 1 \right) \right)} \tag{47}$$

where $x, a, b, \alpha, \beta, \theta > 0$.

The following figure is a plot of the hazard function for some arbitrary parameter values.

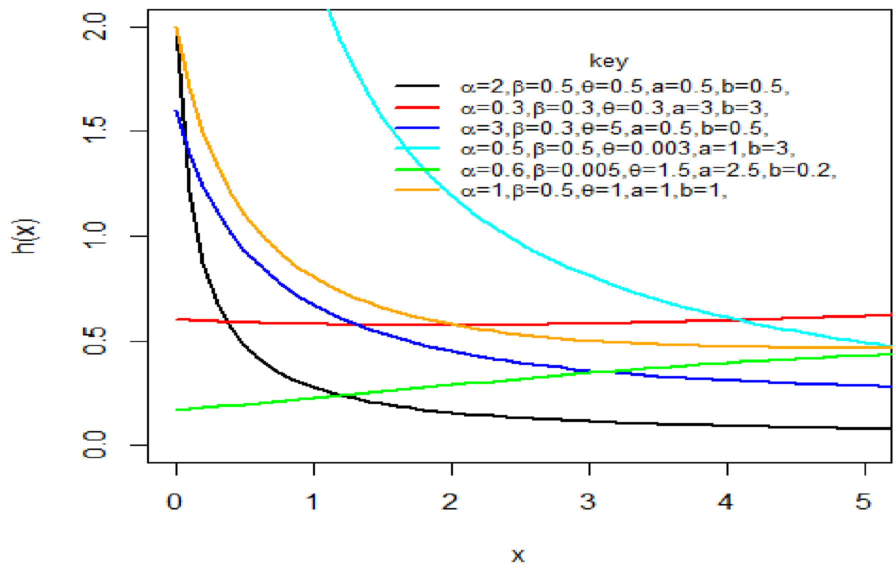

Fig. 4. Hazard function of LOGOMAD

3.4 Quantile function, median and simulation

According to Hyndman and Fan [37], the quantile function for any distribution is defined in the form $Q(u) = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$

To derive the quantile function of the LOGOMAD, F(x) the cdf of the LOGOMAD is considered and inverting it according to the above definition will give us the quantile function of the LOGOMAD as follows:

$$F(x) = 1 - b^a \left\{ b - \log\left[1 - \left(1 - \mathrm{e}^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)} \right) \right] \right\}^a = u \tag{48}$$

Simplifying equation (48) above gives:

$$\mathrm{e}^{\frac{\alpha}{\beta} - b + \left[b^{-a}(1-u)^{-\frac{1}{a}} \right]} = \mathrm{e}^{\theta x + \frac{\alpha}{\beta} e^{\beta x}}$$

$$\frac{\alpha}{\beta} - b + \left[b^{-a}(1-u)^{-\frac{1}{a}} \right] = \theta x + \tfrac{\alpha}{\beta} e^{\beta x}$$

$$\frac{\alpha}{\theta\beta} - \frac{b}{\theta} + \frac{1}{\theta}\left[b^{-a}(1-u)^{-\frac{1}{a}} \right] = x + \tfrac{\alpha}{\theta\beta} e^{\beta x} \tag{49}$$

Using the results by Jodra [38] and Riffi and Hamdan [31], the quantile function of the LOGOMAD is written in a simple form from equation (49) using the principle branch of the Lambert function as:

$$Q(u)=\frac{\alpha}{\theta\beta}-\frac{b}{\theta}+\frac{1}{\theta}\left[b^{-a}(1-u)^{-\frac{1}{a}}\right]-\frac{1}{\beta}W_0\left(\frac{\alpha}{\beta}e^{-b+\left[b^{-a}(1-u)^{-\frac{1}{a}}\right]+\frac{\alpha}{\beta}}\right) \tag{50}$$

where W_0 denotes the principal branch of the Lambert W function, which is briefly described in Jodra [38]. The median of the LOGOMAD distribution can be obtained from the quantile function by substituting $u=0.5$ in Equation (50) which gives:

$$Median=\frac{\alpha}{\theta\beta}-\frac{b}{\theta}+\frac{1}{\theta}\left[b^{-a}(0.5)^{-\frac{1}{a}}\right]-\frac{1}{\beta}W_0\left(\frac{\alpha}{\beta}e^{-b+\left[b^{-a}(0.5)^{-\frac{1}{a}}\right]+\frac{\alpha}{\beta}}\right) \tag{51}$$

Also, random numbers can be generated from the LOGOMAD by setting $Q(u)=X$ and this process is called simulation using inverse transformation method. That means:

$$X=\frac{\alpha}{\theta\beta}-\frac{b}{\theta}+\frac{1}{\theta}\left[b^{-a}(1-u)^{-\frac{1}{a}}\right]-\frac{1}{\beta}W_0\left(\frac{\alpha}{\beta}e^{-b+\left[b^{-a}(1-u)^{-\frac{1}{a}}\right]+\frac{\alpha}{\beta}}\right) \tag{52}$$

The paper presents the quantile based measures of skewness and kurtosis as follows:

Kennedy and Keeping [39] defined the Bowley's measure of skewness based on quartiles as:

$$SK=\frac{Q\left(3/4\right)-2Q\left(1/2\right)+Q\left(1/4\right)}{Q\left(3/4\right)-Q\left(1/4\right)} \tag{53}$$

And Moors [40] presented the Moors' kurtosis based on octiles by:

$$KT=\frac{Q\left(7/8\right)-Q\left(5/8\right)-Q\left(3/8\right)+\left(1/8\right)}{Q\left(6/8\right)-Q\left(1/8\right)} \tag{54}$$

"where $Q(.)$ is calculated by using the quantile function from equation (50).

3.5 Entropy measurement

The Entropy of a distribution is a function that quantifies the uncertainty or randomness in a system or distribution. This sub-section presents the most frequently used measure of entropy called Renyi entropy. The Renyi entropy of a random variable *X* is defined as:

$$I_\zeta(X)=\frac{1}{1-\zeta}\log\int_{-\infty}^{\infty}f^\zeta(x)\,dx \tag{55}$$

for $\zeta>0$ and $\zeta\neq1$.

Now, considering the simplified pdf of the LOGOMAD in equation (29) we get:

$$I_\zeta(X)=\frac{1}{1-\zeta}\log\left[\int_{x=0}^{\infty}\left(\gamma_{k,l,m}\right)^{\zeta}\left(\theta+\alpha e^{\beta x}\right)^{\zeta}x^{\zeta k}e^{\zeta m\beta x}dx\right]$$

$$I_\zeta(X)=\frac{1}{1-\zeta}\log\left[\int_{x=0}^{\infty}\left(\gamma_{k,l,m}\right)^{\zeta}\sum_{p=0}^{\zeta}\binom{\zeta}{p}\theta^{\zeta}\alpha^{p}x^{\zeta k}e^{\beta(p+m\zeta)x}dx\right] \tag{56}$$

Where

and $$\gamma_{k,l,m}=\frac{a}{b}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{-a-1}{k}\binom{k}{l}\binom{l}{m}\left(\frac{\theta}{b}\right)^{k}\left(\frac{\alpha}{b\beta}\right)^{l}(-1)^{l+m}$$

Therefore, solving the integral above and simplifying the result, the Ren'yi entropy of LOGOMAD is obtained as:

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\left(\gamma_{k,l,m}\right)^{\zeta}\sum_{p=0}^{\zeta}\binom{\zeta}{p}\theta^{\zeta}\alpha^{p}\frac{(-1)^{k\zeta+1}\Gamma(2\delta-1)}{\left(\beta(p+m\zeta)\right)^{k\zeta+1}}\right]$$

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\left(\frac{a}{b}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{-a-1}{k}\binom{k}{l}\binom{l}{m}\left(\frac{\theta}{b}\right)^{k}\left(\frac{\alpha}{b\beta}\right)^{l}(-1)^{l+m}\right)^{\zeta}\sum_{p=0}^{\zeta}\binom{\zeta}{p}\theta^{\zeta}\alpha^{p}\frac{(-1)^{k\zeta+1}\Gamma(2\delta-1)}{\left(\beta(p+m\zeta)\right)^{k\zeta+1}}\right] \tag{57}$$

3.6 Order statistics

Suppose $X_1, X_2, \ldots\ldots, X_n$ is a random sample from the LOGOMAD and let $X_{1:n}, X_{2:n}, \ldots\ldots, X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^{k}\binom{n-i}{k}f(x)F(x)^{k+i-1} \tag{58}$$

Using (6) and (7) and simplifying the result, the *pdf* of the i^{th} order statistics$X_{i:n}$, can be expressed from (58) as:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^{k}\binom{n-i}{k}\left[\frac{ab^{a}\left(\theta+\alpha e^{\beta x}\right)}{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^{(a+1)}}\right]\left[\frac{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^{a}-b^{a}}{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^{a}}\right]^{i+k-1} \tag{59}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the LOGOMAD are respectively given by:

$$f_{1:n}(x)=n!\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{ab^a\left(\theta+\alpha e^{\beta x}\right)}{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^{(a+1)}}\right]\left[\frac{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^a-b^a}{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^a}\right]^k \tag{60}$$

and

$$f_{n:n}(x)=n\left[\frac{ab^a\left(\theta+\alpha e^{\beta x}\right)}{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^{(a+1)}}\right]\left[\frac{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^a-b^a}{\left(b+\theta x+\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right)^a}\right]^{n-1} \tag{61}$$

4 Maximum Likelihood Estimation of the Unknown Parameters of the LOGOMAD

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the LOGOMAD with unknown parameters a, b, α, β *and* θ defined previously.

The likelihood function of the LOGOMAD using the pdf in equation (7) is given by:

$$L\left(\underline{X}/a,b,\alpha,\beta,\theta\right)=\left(ab^a\right)^n\prod_{i=1}^{n}\left(\left(\theta+\alpha e^{\beta x_i}\right)\left(b+\theta x_i+\tfrac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)\right)^{-(a+1)}\right) \tag{62}$$

Let the natural logarithm of the likelihood function be, $l(\eta)=\log L\left(\underline{X}\mid a,b,\alpha,\beta,\theta\right)$, therefore, taking the natural logarithm of the function above gives:

$$l(\eta)=n\log a+an\log b+\sum_{i=1}^{n}\log\left(\theta+\alpha e^{\beta x_i}\right)-(a+1)\sum_{i=1}^{n}\log\left(b+\theta x_i+\tfrac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)\right) \tag{63}$$

Differentiating $l(\eta)$ partially with respect to a, b, α, β *and* θ respectively gives the following results:

$$\frac{\partial l(\eta)}{\partial a}=\frac{n}{a}+n\log b-\sum_{i=1}^{n}\log\left(b+\theta x_i+\tfrac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)\right) \tag{64}$$

$$\frac{\partial l(\eta)}{\partial b}=\frac{an}{b}-(a+1)\sum_{i=1}^{n}\left\{\frac{1}{\left(b+\theta x_i+\tfrac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)\right)}\right\} \tag{65}$$

$$\frac{\partial l(\eta)}{\partial \alpha}=\sum_{i=1}^{n}\left\{\frac{e^{\beta x_i}}{\left(\theta+\alpha e^{\beta x_i}\right)}\right\}-\frac{(a+1)}{\beta}\sum_{i=1}^{n}\left\{\frac{e^{\beta x_i}-1}{\left(b+\theta x_i+\tfrac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)\right)}\right\} \tag{66}$$

$$\frac{\partial l(\eta)}{\partial \beta} = \alpha \sum_{i=1}^{n} \left\{ \frac{x_i e^{\beta x_i}}{\left(\theta + \alpha e^{\beta x_i}\right)} \right\} - \frac{\alpha(a+1)}{\beta} \sum_{i=1}^{n} \left\{ \frac{x_i e^{\beta x_i} - \beta^{-1}\left(e^{\beta x_i} - 1\right)}{\left(b + \theta x_i + \frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right)} \right\} \tag{67}$$

$$\frac{\partial l(\eta)}{\partial \theta} = \sum_{i=1}^{n} \left\{ \frac{1}{\left(\theta + \alpha e^{\beta x_i}\right)} \right\} - (a+1) \sum_{i=1}^{n} \left\{ \frac{x_i}{\left(b + \theta x_i + \frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right)} \right\} \tag{68}$$

Making equation (64), (65), (66), (67) and (68) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters $a,\ b,\ \alpha\ ,\ \beta$ *and* θ. However, these solutions cannot be obtained manually except numerically with the aid of suitable statistical software like *R*, *SAS*, *MATHEMATICA e.t.c.*

5 Applications to Real Life Datasets

In this section, we present two real life datasets, their summary and applications. The section fits the proposed distribution (LOGOMAD) together with other two models which include Gompertz-Makeham distribution (*GOMAD*) and the Gompertz distribution (*GOMD*) to the two datasets. The density functions of these distributions are given as follows;

1. Lomax Gompertz-Makeham Distribution (LOGOMAD)

The *pdf* of the LOGOMAD distribution is given as:

$$f(x) = ab^a \frac{\left[\theta + \alpha e^{\beta x}\right] e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}}{\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right] \left\{b - \log\left[1 - \left(1 - e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\right)\right]\right\}^{a+1}} \tag{69}$$

2. Gompertz-Makeham Distribution (GOMAD)

The *pdf* of the GOMAD is given as:

$$f(x) = \left[\theta + \alpha e^{\beta x}\right] e^{-\theta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)} \tag{70}$$

3. Gompertz Distribution (GOMD)

The *pdf* of the GOMD is given as:

$$f(x) = \alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)} \tag{71}$$

In order to evaluate the performance of the models listed above, we used a model selection criterion called Akaike Information Criterion, *AIC.* The formula for this criterion is given as:

$$AIC = -2ll + 2k$$

where ll denotes the log-likelihood value evaluated at the maximum likelihood estimates (MLEs) and k is the number of model parameters. The required computations are carried out using the R package "makLik" which is freely available at URL https://www.R-project.org/ (R Core Team [41]). Our decision rule is that the model with the lowest values of AIC is considered as the best model to fit the data.

Data set I: This dataset comprises of flood data with 20 observations obtained from Dumonceaux and Antle [42] and has been used by Khan et al. [43].

Table 1. Summary Statistics for dataset I

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
20	0.2650	0.3345	0.4070	0.4578	0.4232	0.7400	0.0157	1.0677	0.5999

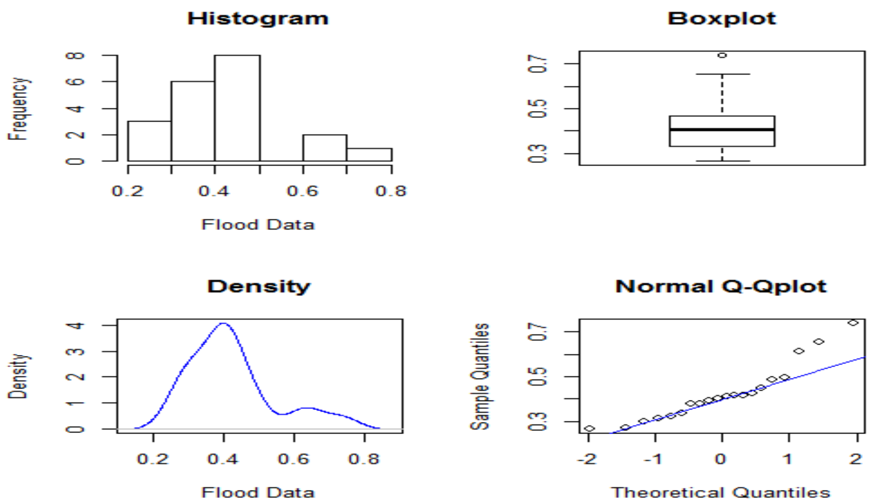

Fig. 5. A graphical summary of dataset I

Based on the descriptive statistics in Table 1 and the histogram, box plot, density and normal Q-Q plot shown in Fig. 5 above, we observed that dataset I is positively skewed.

Table 2. Performance of the fitted distributions using the value of AIC based on dataset I

Distributions	Parameter estimates	log-likelihood value	*AIC*	Rank of models
LOGOMAD	$\hat{\alpha}=$ 0.69217 $\hat{\beta}=$ 4.78422 $\hat{\theta}=$ -2.42827 $\hat{a}=$ 3.42335 $\hat{b}=$ 0.27432	162.6437	-315.2873	**1st**
GOMAD	$\hat{\alpha}=$ -77.397 $\hat{\beta}=$ -6.718 $\hat{\theta}=$ 13.402	135.2997	-264.5993	**2nd**
GOMD	$\hat{\alpha}=$ 0.3041 $\hat{\beta}=$ 6.3888	10.33035	-16.6607	**3rd**

The following figures displayed the histogram and estimated densities with cdfs and Q-Q plots of the fitted models to dataset I.

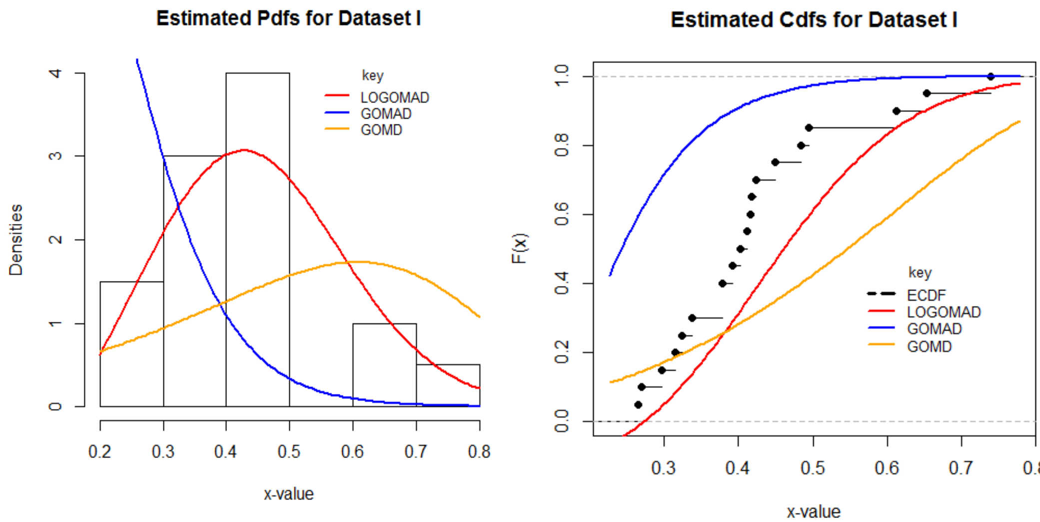

Fig. 6. Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset I

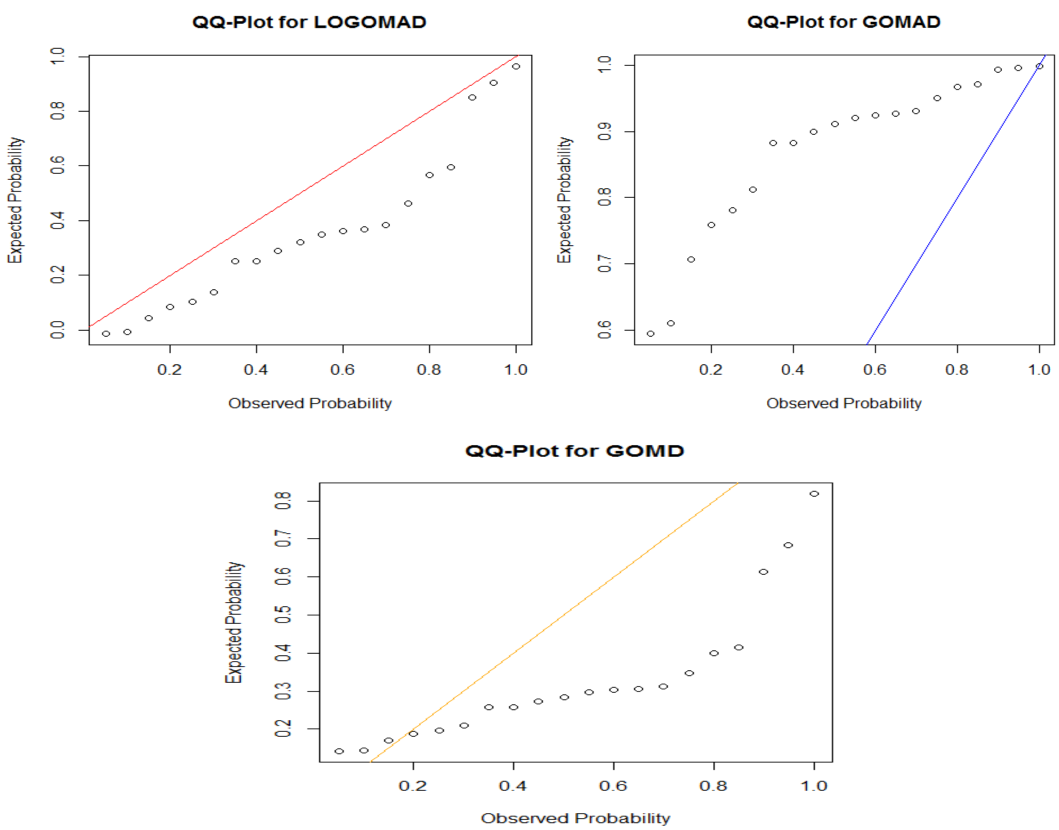

Fig. 7. Probability plots for the fit of the LOGOMAD, GOMAD & GOMD based on dataset I

Dataset II: The second data set represents 66 observations of the breaking stress of carbon fibres of 50mm length (in GPa) given by Nicholas and Padgett [44]. This data set has been used by Cordeiro and Lemonte [45], Al-Aqtash et al. [46], Afify et al. [47], Oguntunde et al. [48], Ieren and Yahaya [49] and Afify et al. [50]. The descriptive statistics for this data are as follows:

Table 3. Descriptive statistics for dataset II

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
66	0.390	2.178	2.835	3.278	2.760	4.900	0.795	-0.1285	3.2230

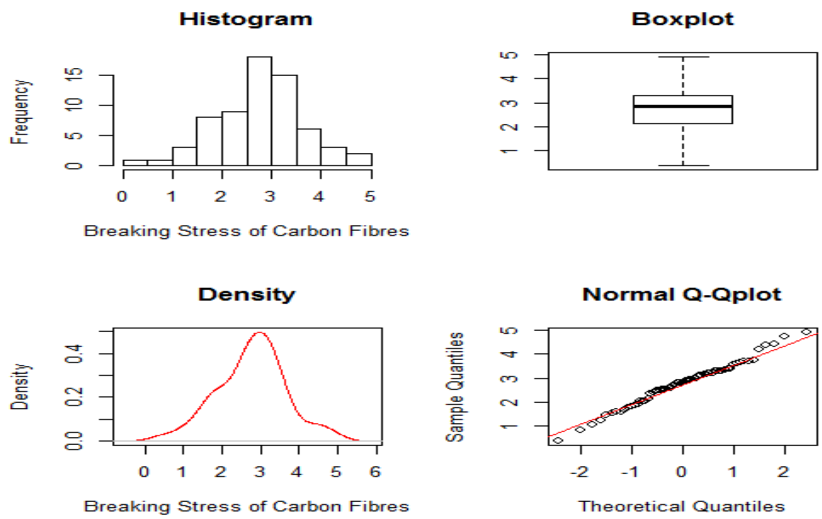

Fig. 8. A graphical summary of Dataset II

Again, the descriptive statistics in Table 3 and the histogram, box plot, density and normal Q-Q plot shown in Fig. 8 above reveal that the second dataset (dataset II) is approximately normal and is not considered to be a skewed dataset.

Table 4. Performance of the fitted distributions using the value of AIC based on dataset II

Distributions	Parameter estimates	log-likelihood value	*AIC*	Rank of models
LOGOMAD	$\hat{\alpha}=$ 5.459e-02 $\hat{\beta}=$ 1.392e+00 $\hat{\theta}=$ -6.341e-02 $\hat{a}=$ 2.217e+00 $\hat{b}=$ 4.200e+00	-85.15788	180.3158	**2nd**
GOMAD	$\hat{\alpha}=$ -3.39869 $\hat{\beta}=$ -0.09616 $\hat{\theta}=$ 3.28506	-94.69054	195.3811	**3rd**
GOMD	$\hat{\alpha}=$ 0.03556 $\hat{\beta}=$ 1.08414	-88.09858	180.1972	**1st**

The figures below displayed the histogram and estimated densities with cdfs and Q-Q plots of the fitted models to dataset II.

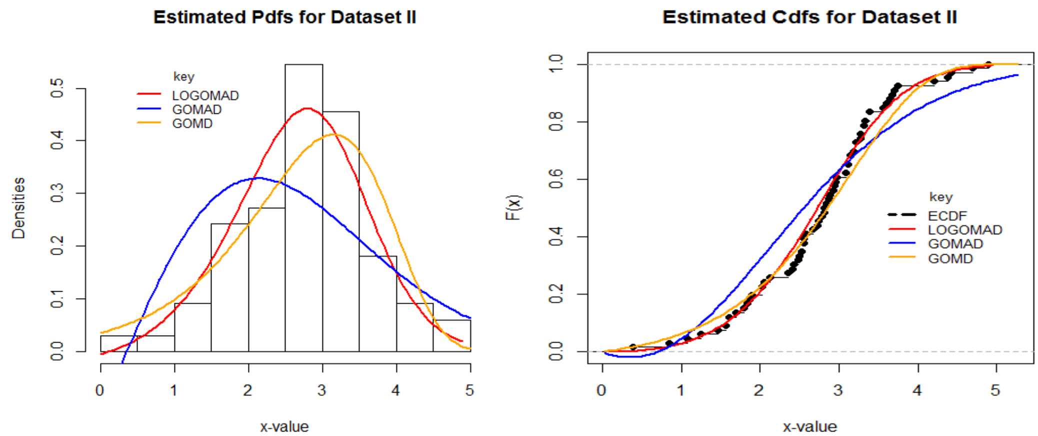

Fig. 9. Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset II

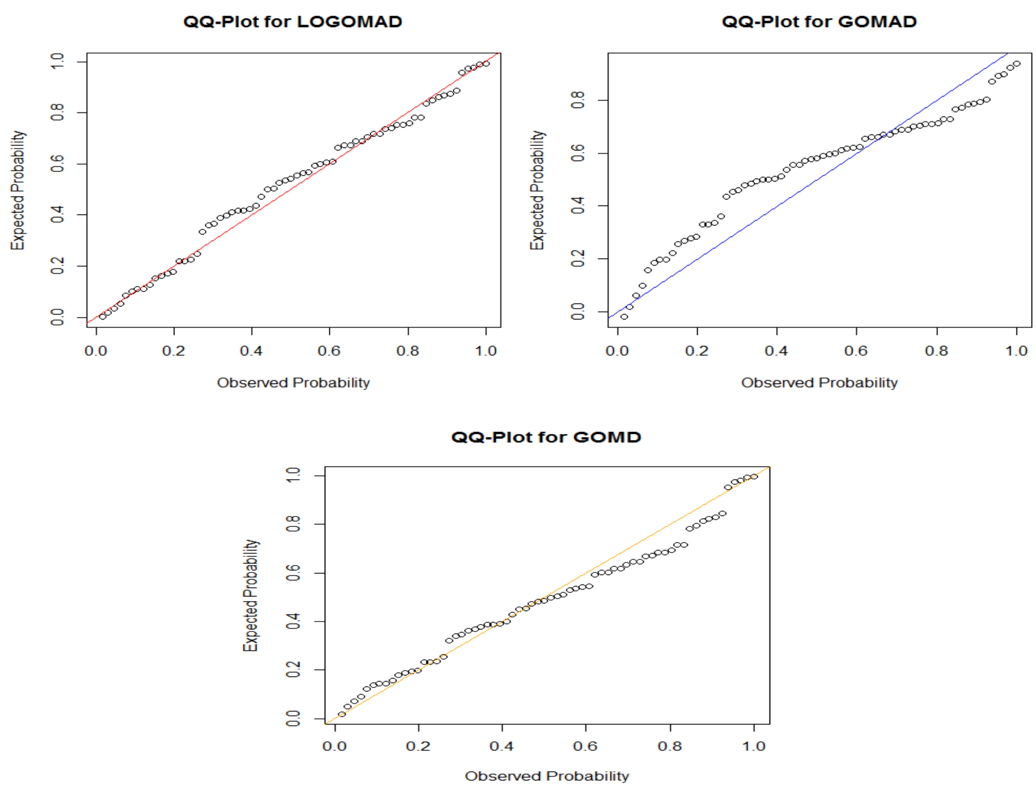

Fig. 10. Probability plots for the fit of the LOGOMAD, GOMAD & GOMD based on dataset II.

Tables 2 and 4 present the parameter estimates and the values of AIC for the LOGOMAD, GOMAD and GOMD using dataset I and dataset II respectively. The values of AIC in Table 2 are lower for the

LOGOMAD compared to the GOMAD and GOMD which is an indication that the Lomax Gompertz-Makeham distribution (LOGOMAD) is more flexible and fits the dataset better than the Gompertz-Makeham distribution (GOMAD) and the conventional Gompertz distribution (GOMD). Also, the histogram with fitted densities and estimated cumulative distribution functions displayed in Fig. 6 for dataset I confirm that the LOGOMAD performs better than the GOMAD and the conventional GOMD. Similarly, the Q-Q plots in figure 7 for dataset I also show that the proposed distribution (LOGOMAD) is more flexible than the other two as shown previously in Tables 2 based on dataset I.

In Table 4, the values of AIC are lower for the GOMD compared to the LOGOMAD and GOMAD which is attributed to the fact that the second dataset (dataset II) is not a skewed data while the proposed model is a skewed distribution based on the plots of the pdf and is not meant for symmetric datasets. However, irrespective of the fact above the histogram with fitted densities and estimated cumulative distribution functions displayed in Fig. 8 as well as the Q-Q plots in Fig. 10 both for dataset II confirm that the LOGOMAD performs better than the GOMAD and the conventional GOMD which is a proof that the LOGOMAD is a flexible model for different kinds of data.

The results above are evidence to the fact that the Lomax generator of distributions by Cordeiro et al. [21] is responsible for the flexibility induced in the Gompertz-Makeham distribution. The results have also shown that the Lomax-G family by Cordeiro et al. [21] should be used to extend other continuous distributions since they are in line with the results of Venegas et al. [33], Omale et al. [34], Ieren et al. [35] as well as Ieren and Kuhe [36].

6 Summary and Conclusion

This article introduced a new extension of the Gompertz-Makeham distribution called Lomax Gompertz-Makeham distribution. It studied the validity and limiting behavior of the new model with some of its mathematical and statistical properties including graphical demonstrations. The study has derived some expressions for moments, moment generating function, characteristics function, quantile function for calculation of median and simulation, the survival function, hazard function, Rènyi entropy and density for distribution of minimum and maximum order statistics with appropriate discussions. The unknown model parameters have been estimated using the method of maximum likelihood estimation. Some plots of the distribution revealed that it is positively skewed and that its shape varies depending on the values of the parameters. The implications of the plots for the survival function indicate that the Lomax Gompertz-Makeham distribution could be used to model age-dependent or time-dependent events or variables whose survival decreases as time grows or where survival rate decreases with increase in age or time. Also, the hazard rate of the Lomax Gompertz-Makeham distribution is decreasing which is useful for most real life situations. The results of the applications of the Lomax Gompertz-Makeham distribution to two real life datasets show that the proposed distribution is more flexible compared to the Gompertz-Makeham distribution and conventional Gompertz distribution and therefore, we are hopeful that this new extension of the Gompertz-Makeham distribution will be applied in modeling real life situations especially in the areas of survival analysis, modeling human mortality, constructing actuarial tables and development growth models.

Competing Interests

Authors have declared that no competing interests exist.

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