



## Analysis of Two Finite Difference Schemes for a Channel Flow Problem

Chinedu Nwaigwe<sup>1\*</sup> and Azubuiké Weli<sup>1</sup>

<sup>1</sup>Department of Mathematics, Rivers State University, Port Harcourt, Nigeria.

### Authors contributions

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

### Article Information

DOI: 10.9734/ARJOM/2019/v15i430156

*Editor(s):*

(1) Dr. Wei-Shih Du, Professor, Department of Mathematics, National Kaohsiung Normal University, Taiwan.

*Reviewers:*

(1) Augusto César Albuquerque Ferreira, University of Salamanca, Spain.  
(2) Volodymyr Grimalsky, Autonomous University of State Morelos (UAEM), Mexico.

Complete Peer review History: <http://www.sdiarticle4.com/review-history/52512>

*Received: 26 August 2019*

*Accepted: 01 November 2019*

*Published: 08 November 2019*

**Original Research Article**

## Abstract

**Aims/ Objectives:** To investigate the influence of a model parameter on the convergence of two finite difference schemes designed for a convection-diffusion-reaction equation governing the pressure-driven flow of a Newtonian fluid in a rectangular channel.

**Methodology:** By assuming a uni-directional and incompressible channel flow with an exponentially time-varying suction velocity, we formulate a variable-coefficient convection-diffusion-reaction problem. In the spirit of the method of manufactured solutions, we first obtain a benchmark analytic solution via perturbation technique. This leads to a modified problem which is exactly satisfied by the benchmark solution. Then, we formulate central and backward difference schemes for the modified problem. Consistency and convergence results are obtained in detail. We show, theoretically, that the central scheme is convergent only for values of a model parameter up to an upper bound, while the backward scheme remains convergent for all values of the parameter. An estimate of this upper bound, as a function of the mesh size, is derived. We then conducted numerical experiments to verify the theoretical results.

**Results:** Numerical results showed that no numerical oscillations were observed for values of the model parameter less than the theoretically derived bound.

**Conclusion:** We therefore conclude that the theoretical bound is a safe value to guarantee non-oscillatory solutions of the central scheme.

\*Corresponding author: E-mail: [nwaigwe.chinedu@ust.edu.ng](mailto:nwaigwe.chinedu@ust.edu.ng);

*Keywords:* Finite difference schemes; convergence analysis; consistency analysis; perturbation method; pressure-driven channel flow; experimental order of convergence; method of manufactured solutions.

**2010 Mathematics Subject Classification:** 53C25; 83C05; 57N16.

## 1 Introduction

Let  $\epsilon, h, t_0, T, k \in \mathbb{R}, (z, t) \in \mathbb{R} \times (\mathbb{R}^+ \cup \{0\})$  and  $u_* : \mathbb{R} \rightarrow \mathbb{R}$ . We consider the following problem: find the unknown  $u(z, t)$  such that:

$$\begin{aligned} \frac{\partial u(z, t)}{\partial t} + \epsilon f(t) \frac{\partial}{\partial z} u(z, t) &= \frac{\partial^2}{\partial z^2} u(z, t) - P_x, \quad z \in (0, h), t \in (0, T), \\ u(0, t) = u(h, t) &= 0, \quad \forall t \in [0, T], \\ u(z, 0) &= u_*(z), \quad \forall z \in [0, h], \end{aligned} \tag{1.1}$$

where  $0 \leq |f(t)| \leq f_{max} < \infty$ , for all  $t \in [0, T]$ . In this study, we consider

$$f(t) = e^{-kt}. \tag{1.2}$$

$$\tag{1.3}$$

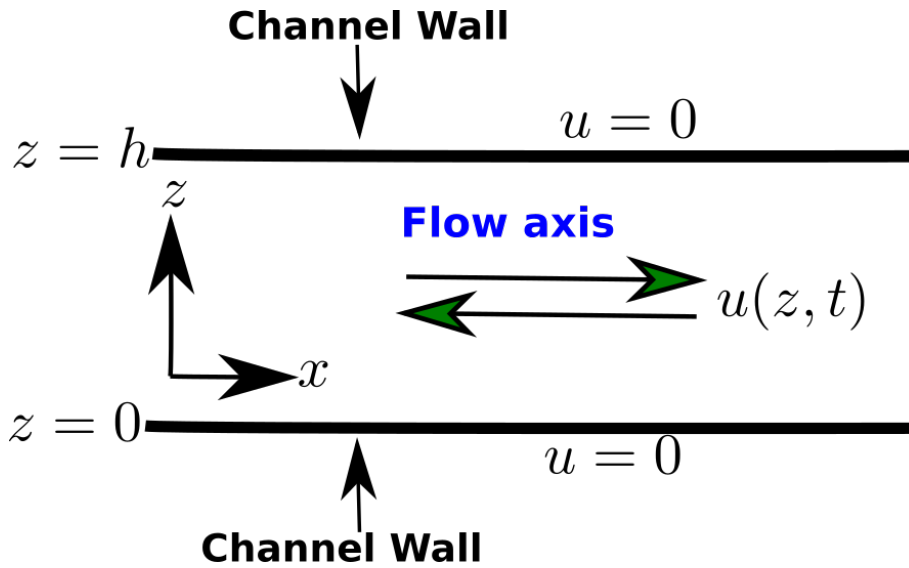


Fig. 1. Physical set up of flow in a horizontal channel

The above problem (1.1)-(1.2) governs the flow of a viscous fluid in a horizontal rectangular channel with vertical dimension  $h$  and constant pressure gradient,  $P_x$ , along the horizontal direction, see figure 1. The velocity components are  $(u(z, t), 0, \epsilon f(t))$ , and the problem is thus to find the only unknown velocity component,  $u$ . Problems of this type have been extensively applied to fluid flows with and without heat and/or mass transfer, see [1, 2, 3, 4, 5, 6, 7]. Most of these studies have adopted approximate analytic methods which are only valid for small values of relevant model parameter(s). Also, the studies that adopted numerical methods have not focused on understanding

the influence of the smallness or largeness of the model parameters on the properties of the considered numerical schemes.

Numerical analysis of diffusion-reaction with and without convection have also been studied widely. It is well known that traditional discretization methods face difficulties when applied to convection-diffusion problems, especially when convection dominates diffusion. Consequently, different methods have been proposed to study these problems. Morosanu et al. [8] investigate the error and stability analyses of a nonlinear diffusion-reaction problem. Three different approaches for approximating nonlinearities were discussed and the convergence results obtained.

Mickens [9] constructed nonstandard finite difference scheme for nonlinear diffusion-reaction models with reaction terms that are polynomial functions of the dependent variables. The constructed schemes are positivity preserving, and the authors also provided rules for constructing such schemes. This work was extended in [10] to convection-diffusion-reaction (CDR) problems with linear convection, and the Fischer equation is utilized for illustrating the method. Mingrong [11] also proposed a method for splitting nonlinear reaction terms leading to a modified upwind method that is second-order accurate in space; error estimate in  $l^\infty$ -norm was obtained.

James and co-workers [12] developed a method based on the unifying Eulerian-Lagrangian particle scheme and operator splitting approach. In [13], a positive preserving explicit scheme is developed for CDR problems with constant advection coefficients and reaction terms that are sum of positive and negative functions of the solution variable; a strategy for the generalization is presented. A draw back of the scheme is that it is not unconditionally consistent, hence requires complicated procedure to fix the consistency issues [14, 15]. Ndivhuwo [14] investigated different numerical schemes for linear and nonlinear convection-diffusion-reaction models. Consistency, stability and spectral analysis were conducted for the schemes. It was found that the standard upwind schemes and a nonstandard upwind schemes are both consistent while the unconditionally positive finite difference scheme [13] is not. In [16], a class of exact finite difference schemes are proposed and used to derive nonstandard implicit schemes which preserve nonnegativity and boundedness of exact solutions of the continuous problem. A review of nonstandard finite difference methods can be found in [17], see also [18] for another nonstandard scheme that is stable, implicit and three-level. Other related work on CDR can be found in [19, 20].

The methods outlined above target some special properties of analytical solutions, especially the non-negativity issue, and they are more complicated to apply than corresponding standard methods such as the upwind scheme [21, 22]. In fluid dynamics problems in which the sought solution is the fluid velocity, we do not really care about the sign of computed solutions, hence standard methods, if properly applied, may be adequate.

Moreso, fluid dynamics models (and others too) usually include some parameters of interest. These parameters may be part of the coefficients of the diffusive (viscous) or convective (inertia) terms. For small values of such model parameters, such as  $\epsilon$ , an approach to obtain an analytic solution is via regular perturbation method, see [3, 23] for examples. And indeed, such obtained solutions reasonably approximate the exact solutions provided the parameter remains small. However, when the parameter becomes large such analytic solutions hugely deviate from the true solution of the problem. Bearing this effect (on perturbation methods) in mind, we are curious to know if such effects may also exist for numerical methods. In particular, we want to know if and how the values of the parameter affect the convergence of numerical methods designed for problem (1.1)-(1.2), so that one can understand when these methods are being properly applied to this model. We investigate this for two standard finite-difference schemes being one central scheme and one backward scheme. Note: the parameter of interest in this study is  $\epsilon$ .

In section 2, we use the perturbation method to derive an analytic solution (here called perturbation

solution) for problem (1.1)-(1.2). We then present the error term or residual with which the perturbation solution fails to satisfy problem (1.1)-(1.2). This then leads to a modified model which is exactly satisfied (for all values of  $\epsilon$ ) by the obtained perturbation solution. In section 3, we present the two schemes and analyse them for error and convergence in section 3.2. The numerical results are presented and discussed in section 4 and the paper is concluded in section 5.

## 2 Analytical Solution By Perturbation Method

We assume  $\epsilon$  to be small, hence propose an analytical solution using perturbation method. To this end, we assume a solution of the form:

$$u(z, t) = u_0(z) + \epsilon e^{-kt} u_1(z). \quad (2.1)$$

Substituting into (1.1), the order one,  $O(1)$ , terms give the problem:

$$\begin{aligned} \frac{d^2 u_0}{dz^2} &= P_x, \\ u_0(0) &= u_0(h) = 0, \end{aligned} \quad (2.2)$$

while the  $O(\epsilon)$  terms give the following problem:

$$\begin{aligned} \frac{d^2 u_1}{dz^2} + k u_1 &= \frac{d u_0}{dz}, \\ u_1(0) &= u_1(h) = 0. \end{aligned} \quad (2.3)$$

Solving (2.2)-(2.3) and substituting into (2.1), we obtain the following approximate solution for (1.1):

$$u(z, t) = \frac{P_x}{2} z(z - h) + \epsilon e^{-kt} \left( A \cos(\sqrt{k}z) + B \sin(\sqrt{k}z) + \frac{P_x}{k} \left( z - \frac{h}{2} \right) \right), \quad (2.4)$$

where  $A = \frac{P_x h}{2k}$ , and  $B = -\frac{P_x h (1 + \cos(h\sqrt{k}))}{2k \sin(h\sqrt{k})}$ .

### 2.1 The modified problem

It is easy to show that the analytic solution (2.4) satisfies the model problem (1.1) with a residual,  $R(z, t)$  given by

$$R(z, t) = \epsilon^2 \sqrt{k} e^{-2kt} \left( B \cos(\sqrt{k}z) - A \sin(\sqrt{k}z) + \frac{P_x}{k\sqrt{k}} \right) = O(\epsilon^2) \quad (2.5)$$

which vanishes as soon as  $\epsilon \rightarrow 0$ . One can also observe that the solution (2.4) satisfies the steady state solution of (1.1) exactly, namely

$$u_s(z, t) := \lim_{t \rightarrow \infty} u(z, t) = \frac{P_x}{2} z(z - h), \quad (2.6)$$

which is the exact steady-state solution of problem (1.1).

With the above residual, it is easy to verify that the solution (2.4) is an exact solution of the following modified problem:

$$\begin{aligned} \frac{\partial u(z, t)}{\partial t} + \epsilon e^{-kt} \partial_z u(z, t) &= \partial_{zz}^2 u(z, t) - P_x + R(z, t), \quad (z, t) \in (0, h) \times (t_0, T), \\ u(0, t) &= u(h, t) = 0, \quad \forall t \geq 0, \\ u(z, 0) &= \frac{P_x}{2} z(z - h) + A \cos(\sqrt{k}z) + B \sin(\sqrt{k}z) \\ &+ \frac{P_x}{k} \left( z - \frac{h}{2} \right) \forall z \in [a, b]. \end{aligned} \quad (2.7)$$

The above problem shall be referred to as the Modified Problem.

### 3 Numerical Schemes for the Modified Problem

We shall formulate the numerical schemes for (2.7) instead of (1.1), so that we can use (2.4) as a benchmark for the numerical experiments. This is in line with the method of manufactured solutions [24, 25] for verifying numerical schemes and codes.

Let  $M \in \mathbb{Z}^+$ ,  $S_I := \{0, 1, 2, 3, \dots, M\}$  and  $S_N := \{0, 1, 2, \dots\}$ . Define  $\Delta z := \frac{h}{M}$  and  $\Delta t$  be given. We discretize the domain,  $z_i = i\Delta z \forall i \in S_I$ , and in time  $t^n = n\Delta t \forall n \in S_N$ . We also define  $c_i^n \approx c(x_i, t^n)$ . We consider the following two numerical schemes:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \epsilon e^{-kt_{n+1}} \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta z} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta z)^2} - P_x + R(z_i, t_{n+1}) \quad (3.1)$$

and

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \epsilon e^{-kt_{n+1}} \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta z} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta z)^2} - P_x + R(z_i, t_{n+1}) \quad (3.2)$$

which we call, here, the **central** and **backward** schemes respectively. We will investigate the above schemes by comparing their accuracies and convergence for different values of  $\epsilon$ . In particular, our primary goal is to understand how the consistency and convergence of the above schemes depend on the parameter,  $\epsilon$ .

#### 3.1 Error analysis

We define the truncation error,  $T_{c,i}^n$  for the central scheme (3.1), as follows:

$$T_{c,i}^n = \frac{u(z_i, t^{n+1}) - u(z_i, t^n)}{\Delta t} + \epsilon e^{-kt^{n+1}} \frac{u(z_{i+1}, t^{n+1}) - u(z_{i-1}, t^{n+1})}{2\Delta z} - \frac{u(z_{i+1}, t^{n+1}) - 2u(z_i, t^{n+1}) + u(z_{i-1}, t^{n+1})}{(\Delta z)^2} + P_x - R(z_i, t^{n+1}),$$

and we also define the truncation error,  $T_{b,i}^n$  for the backward scheme (3.2), as follows:

$$T_{b,i}^n = \frac{u(z_i, t^{n+1}) - u(z_i, t^n)}{\Delta t} + \epsilon e^{-kt^{n+1}} \frac{u(z_i, t^{n+1}) - u(z_{i-1}, t^{n+1})}{\Delta z} - \frac{u(z_{i+1}, t^{n+1}) - 2u(z_i, t^{n+1}) + u(z_{i-1}, t^{n+1})}{(\Delta z)^2} + P_x - R(z_i, t^{n+1}).$$

**Theorem 3.1** (Consistency of Central Scheme). *The truncation error,  $T_{c,i}^n$  satisfies:*

$$|T_{c,i}^n| \leq \frac{\Delta t}{2} M_{tt} + \frac{(\Delta z)^2}{6} (2\epsilon f_{max} M_{zzz} + M_{zzzz}) \quad (3.3)$$

for all  $n \in S_N; i \in S_I$ ,

where  $f_{max} := \max |e^{-kt}|$ ,  $M_{tt} := \max |u_{tt}(z, t)|$ ,  $M_{zzz} := \max |u_{zzz}(z, t)|$  and  $M_{zzzz} := \max |u_{zzzz}(z, t)|$  taking all over  $(z, t) \in [0, 1] \times [0, T]$ . Note that subscript,  $z$ , indicates partial derivatives.

*Proof.* By the Taylor's theorem:

$$\begin{aligned} \frac{u(z_i, t^{n+1}) - u(z_i, t^n)}{\Delta t} &= u_t(z_i, t^{n+1}) + \frac{\Delta t}{2} u_{tt}(z_i, \rho^{n+1}), \quad \rho^{n+1} \in (t^n, t^{n+1}). \\ \frac{u(z_{i+1}, t^{n+1}) - u(z_{i-1}, t^{n+1})}{2\Delta z} &= u_x(z_i, t^{n+1}) + \frac{(\Delta z)^2}{6} (u_{zzz}(\nu_l, t^{n+1}) + u_{zzz}(\nu_r, t^{n+1})), \\ \nu_l &\in (z_{i-1}, z_i), \nu_r \in (z_i, z_{i+1}). \\ \frac{u(z_{i+1}, t^{n+1}) - 2u(z_i, t^{n+1}) + u(z_{i-1}, t^{n+1})}{(\Delta z)^2} &= u_{zz}(z_i, t^{n+1}) \\ &+ \frac{(\Delta z)^2}{12} (u_{zzzz}(\mu_l, t^{n+1}) + u_{zzzz}(\mu_r, t^{n+1})), \quad \mu_l \in (z_{i-1}, z_i), \mu_r \in (z_i, z_{i+1}). \end{aligned}$$

Hence,

$$\begin{aligned} T_{c,i}^n &= u_t(z_i, t^{n+1}) + \epsilon e^{-kt^{n+1}} u_z(z_i, t^{n+1}) - u_{zz}(z_i, t^{n+1}) + P_x - R(z_i, t_{n+1}) \\ &= \frac{\Delta t}{2} u_{tt}(z_i, \rho^{n+1}) + \epsilon e^{-kt^{n+1}} \frac{(\Delta z)^2}{6} (u_{zzz}(\nu_l, t^{n+1}) + u_{zzz}(\nu_r, t^{n+1})) \\ &\quad - \frac{(\Delta z)^2}{12} (u_{zzzz}(\mu_l, t^{n+1}) + u_{zzzz}(\mu_r, t^{n+1})) \\ &\leq \frac{\Delta t}{2} u_{tt}(z_i, \rho^{n+1}) + \epsilon e^{-kt^{n+1}} \frac{(\Delta z)^2}{6} (u_{zzz}(\nu_l, t^{n+1}) + u_{zzz}(\nu_r, t^{n+1})) \\ &\quad + \frac{(\Delta z)^2}{12} (M_{zzzz} + M_{zzzz}) \\ &\leq \frac{\Delta t}{2} M_{tt} + \epsilon f_{max} \frac{(\Delta z)^2}{6} 2M_{zzz} + \frac{(\Delta z)^2}{6} M_{zzzz} \\ &= \frac{\Delta t}{2} M_{tt} + \frac{(\Delta z)^2}{6} (2\epsilon f_{max} M_{zzz} + M_{zzzz}). \end{aligned}$$

Hence, the result.  $\square$

**Theorem 3.2** (Consistency of Backward Scheme). *The truncation error,  $T_{b,i}^n$  of the backward scheme (3.2) satisfies:*

$$\begin{aligned} |T_{b,i}^n| &\leq \frac{\Delta t}{2} M_{tt} + \epsilon f_{max} \frac{\Delta z}{2} M_{zz} + O((\Delta z)^2). \\ &\text{for all } n \in S_N; i \in S_I, \end{aligned} \tag{3.4}$$

where  $f_{max} := \max |e^{-kt}|$ ,  $M_{tt} := \max |u_{tt}(z, t)|$  and  $M_{zz} := \max |u_{zz}(z, t)|$  taking all over  $(z, t) \in [0, 1] \times [0, T]$ .

*Proof.* By the Taylor's theorem:

$$\begin{aligned} \frac{u(z_i, t^{n+1}) - u(z_{i-1}, t^{n+1})}{\Delta z} &= u_z(z_i, t^{n+1}) + \frac{(\Delta z)}{2} u_{zz}(s_i, t^{n+1}), \\ s_i &\in (z_{i-1}, z_i). \end{aligned}$$

Hence,

$$\begin{aligned} T_{b,i}^{n+1} &= u_t(z_i, t^{n+1}) + \epsilon e^{-kt^{n+1}} u_z(z_i, t^{n+1}) - u_{zz}(z_i, t^{n+1}) + P_x + R(z_i, t_{n+1}) \\ &= \frac{\Delta t}{2} u_{tt}(z_i, \rho^{n+1}) - \epsilon e^{-kt^{n+1}} \frac{\Delta z}{2} u_{zz}(s_i, t^{n+1}) + O((\Delta z)^2) \\ &\leq \frac{\Delta t}{2} M_{tt} + \epsilon f_{max} \frac{\Delta z}{2} M_{zz} + O((\Delta z)^2). \end{aligned}$$

Hence, the result.  $\square$

### 3.2 Convergence analysis

Define the following:

$$A_0 = \frac{\Delta t}{2\Delta z} \epsilon e^{-kt^{n+1}}, \quad \mu_0 = \frac{\Delta t}{(\Delta z)^2}. \quad (3.5)$$

**Theorem 3.3** (Convergence of Central Scheme). *Let  $\bar{u}(z_i, t^n)$  be the exact solution of (2.7) and  $u_i^n$  be the numerical solution computed with the scheme (3.1). Assuming that*

$$\epsilon \leq \frac{2}{\Delta z f_{max}}, \quad (3.6)$$

then

$$\max_{1 \leq i \leq N-1} |\bar{u}(z_i, t^n) - u_i^n| \leq T \left[ \frac{\Delta t}{2} M_{tt} + \frac{(\Delta z)^2}{6} (2\epsilon f_{max} M_{zzz} + M_{zzzz}) \right]$$

for all  $n \in S_N$ .

*Proof.* Define the error  $e_i^n = \bar{u}(z_i, t^n) - u_i^n$  and the maximum error,  $E^n = \max_i |e_i^n|$ . The central scheme (3.1) can be written as

$$(-A_0 - \mu_0)u_{i-1}^{n+1} + (1 + 2\mu_0)u_i^{n+1} + (A_0 - \mu_0)u_{i+1}^{n+1} = u_i^n - \Delta t(P_x - R_i^{n+1}).$$

Combining this with the truncation error equation give the following error equation:

$$\begin{aligned} (1 + 2\mu_0)e_i^{n+1} &= (A_0 + \mu_0)e_{i-1}^{n+1} + (\mu_0 - A_0)e_{i+1}^{n+1} + e_i^n + \Delta t T_{c,i}^n \\ &\leq |(A_0 + \mu_0)||e_{i-1}^{n+1}| + |(\mu_0 - A_0)||e_{i+1}^{n+1}| + |e_i^n| + |\Delta t T_{c,i}^n|. \end{aligned} \quad (3.7)$$

The inequality (3.6) implies

$$\epsilon f_{max} \frac{\Delta t}{2\Delta z} \leq \frac{\Delta t}{(\Delta z)^2},$$

that is,

$$A_0 = \epsilon e^{-kt^{n+1}} \frac{\Delta t}{2\Delta z} \leq \epsilon f_{max} \frac{\Delta t}{2\Delta z} \leq \frac{\Delta t}{(\Delta z)^2} = \mu_0.$$

Hence, the inequality (3.7) becomes:

$$\begin{aligned} (1 + 2\mu_0)e_i^{n+1} &\leq (A_0 + \mu_0)|e_{i-1}^{n+1}| + (\mu_0 - A_0)|e_{i+1}^{n+1}| + |e_i^n| + |\Delta t T_{c,i}^n| \\ &\leq (A_0 + \mu_0)E^{n+1} + (\mu_0 - A_0)E^{n+1} + E^n + \Delta t |T_{c,i}^n| \\ &\leq 2\mu_0 E^{n+1} + E^n + \Delta t T_{c,max}. \end{aligned}$$

Taking maximum over  $i$ , we have

$$E^{n+1} \leq E^n + \Delta t T_{c,max} \leq E^0 + (n+1)\Delta t T_{c,max}$$

or

$$\begin{aligned} E^n &\leq E^0 + n\Delta t T_{c,max} = n\Delta t T_{c,max} \leq TT_{c,max} \\ &\leq T \left[ \frac{\Delta t}{2} M_{tt} + \frac{(\Delta z)^2}{6} (2\epsilon f_{max} M_{zzz} + M_{zzzz}) \right] \quad (\text{by theorem 3.1 above}). \end{aligned}$$

□

**Remark 3.1.** The inequality (3.6) gives a theoretical upper bound on  $\epsilon$  for the central scheme (3.1) to be convergent.

**Theorem 3.4** (Convergence of Backward Scheme). Let  $\bar{u}(z_i, t^n)$  be the exact solution of (2.7) and  $u_i^n$  be the numerical solution computed with the scheme (3.2), then

$$\max_{1 \leq i \leq N-1} |\bar{u}(z_i, t^n) - u_i^n| \leq T \left( \frac{\Delta t}{2} M_{tt} + \epsilon f_{max} \frac{\Delta z}{2} M_{zz} + O((\Delta z)^2) \right)$$

for all  $n \geq 1$ .

*Proof.* Define the error  $e_i^n = \bar{u}(z_i, t^n) - u_i^n$  and the maximum error,  $E^n = \max_i |e_i^n|$ . The backward scheme (3.2) can be written as

$$(-A_0 - \mu_0)u_{i-1}^{n+1} + (1 + A_0 + 2\mu_0)u_i^{n+1} - \mu_0 u_{i+1}^{n+1} = u_i^n - \Delta t(P_x - R_i^{n+1}).$$

Combining this with the truncation error equation give the following error equation:

$$\begin{aligned} (1 + A_0 + 2\mu_0)e_i^{n+1} &= (A_0 + \mu_0)e_{i-1}^{n+1} + \mu_0 e_{i+1}^{n+1} + e_i^n + \Delta t T_{b,i}^n \\ &\leq |(A_0 + \mu_0)||e_{i-1}^{n+1}| + |\mu_0||e_{i+1}^{n+1}| + |e_i^n| + |\Delta t T_{b,i}^n| \\ &\leq (A_0 + 2\mu_0)E^{n+1} + E^n + \Delta t T_{b,max}. \end{aligned}$$

Taking maximum over  $i$ , we have

$$\begin{aligned} E^n &\leq E^0 + n\Delta t T_{b,max} = n\Delta t T_{b,max} \leq T T_{b,max} \\ &\leq T \left( \frac{\Delta t}{2} M_{tt} + \epsilon f_{max} \frac{\Delta z}{2} M_{zz} + O((\Delta z)^2) \right) \quad (\text{by theorem 3.2 above}). \end{aligned}$$

□

Hence this scheme converges for all values of the model parameter.

**Remark 3.2.** By setting  $R(x_i, t^{n+1})$  to zero in the two schemes, we have that all the above theoretical results are also true for problem (1.1).

## 4 Numerical Results

We now present some numerical experiments for the numerical schemes discussed above. When not specified, we investigate the results with the following data:  $h = 0.2, k = 0.5, P_x = 1.0$  and for various values of  $\epsilon$ .

### 4.1 Experimental order of convergence (EOC)

We numerically verify the theoretical results obtained for the order of convergence of the presented schemes. This is done via an experimental order of convergence (eoc) study, and the goal is to numerically verify the theoretical derived order of accuracy of the schemes. The formula for the eoc can be found in [1], see also [26]. For this purpose, we use  $\Delta t = 0.005, \epsilon = 100.0, k = 0.5, P_x = 1.0, h = 0.2$  and for various numbers of grid points, namely  $N_{pts} = 3 \times 2^j$  for  $j = 0, 1, 2, \dots, 11$ . The results for each grid spacing is outputted at time  $t = 0.5$ , and the errors (in 2-norm) are computed. The errors and computed eoc for the both schemes are displayed in table 4.1. The results confirm that the central and backward schemes have spatial accuracy of second and first orders, respectively, as theoretically obtained.



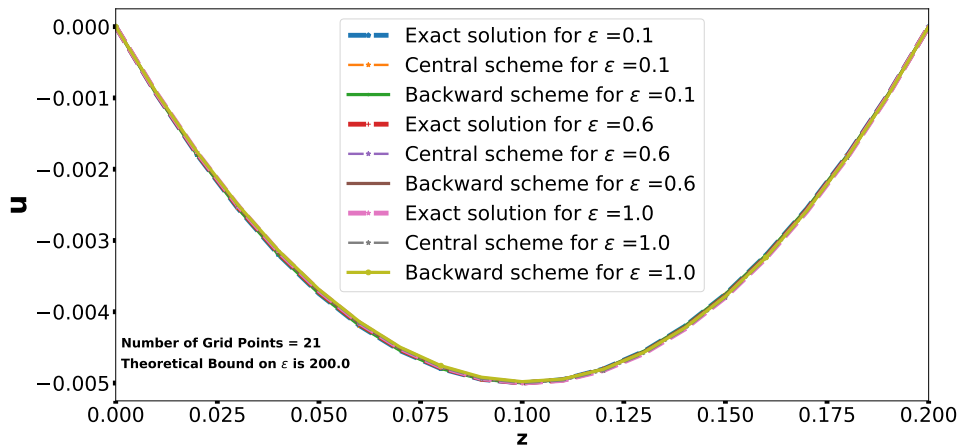
**Table 1. Experimental Order of Convergence**

$N_{pt}$	$Error_{central}$	$EOC_{central}$	$Error_{backward}$	$EOC_{backward}$
3	0.0160733830815	-	0.00201165875699	-
6	0.00103402234412	3.958334349258829	0.00102656371945	0.9705624185080126
12	0.000195312455426	2.404411500567002	0.00052234246	0.9747552923027915
24	4.37756536488e-05	2.157583329463647	0.000264277857679	0.9829404268652807
48	1.04336838914e-05	2.0688800909313216	0.000133127379183	0.9892482462607187
96	2.55058205972e-06	2.032350207574197	6.6851013295e-05	0.9937859754012208
192	6.30468800041e-07	2.01632963431968	3.35034993538e-05	0.9966376389468367
384	1.56441552848e-07	2.010801216255631	1.67719316502e-05	0.998262933233761
768	3.8666953969e-08	2.016450737378392	8.39091320949e-06	0.9991491186068366
1536	9.3211912107e-09	2.052514876648072	4.19649071027e-06	0.9996444451184606
3072	2.03359980337e-09	2.196478541317043	2.09828760576e-06	0.9999709499361441
6144	4.76141336684e-10	2.094574007513257	1.04893637694e-06	1.0002852630539303

## 4.2 Presence of oscillatory solutions

We investigate the schemes for the presence of numerical oscillations using various values of  $\epsilon$ .

Figure 2 shows the results for small values of  $\epsilon$  ( $0 < \epsilon \leq 1$ ) using only 21 numerical grid points. We make the following observations (i) The numerical solutions (both) highly agree with the exact solution for all values of  $\epsilon$ , and (ii) The two schemes compute very close results. These results are replicated in figure 3. which are obtained using 101 grid points and for the same values of  $\epsilon$ . The conclusion from these experiments is that for small values of  $\epsilon$ , the numerical schemes are very highly accurate even with very small number of grid points.



**Fig. 2. Small values of the parameter using 21 grid points**

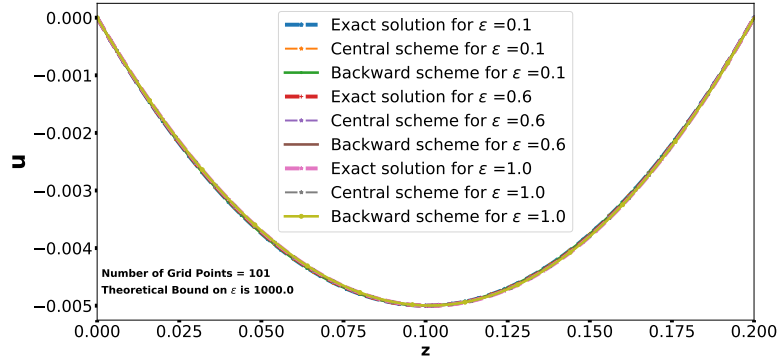


Fig. 3. Small values of the parameter using 101 grid points

Next, we increase the values of the model parameter to 10, 50 and 200. The results for 21 grid points are shown in figure 4. We can see that the numerical schemes experience more error than when  $\epsilon$  is small. In particular, the backward scheme is much less accurate than the central scheme, and the errors increase as  $\epsilon$  increases, keeping the number of grid points is kept constant - it is worst for  $\epsilon = 200$ . Now, we increase the grid points to 101 in figure 5. and observe that the errors in the numerical solutions reduce drastically, especially for the central scheme. For the central scheme, the theoretically estimated threshold for  $\epsilon$  using this grid spacing is 1000 which is far more than the values used in the experiments reported in this figure . We can see that no instability is observed for the central scheme. We also tested other values of  $\epsilon$  and found that the central scheme will never produce oscillatory solutions as long as  $\epsilon$  is less than its threshold of 1000. Infact, we observed that even with values of  $\epsilon$  more than 1000 it the central scheme still computes non-oscillatory solutions. The oscillations only appear when  $\epsilon$  is very much larger than the threshold (multiples of it). So, 1000 is a safe upper bound to guarantee non-oscillatory solutions for the central scheme. The numerical results show that the backward scheme never produces oscillatory solutions no matter how large the value of  $\epsilon$ .

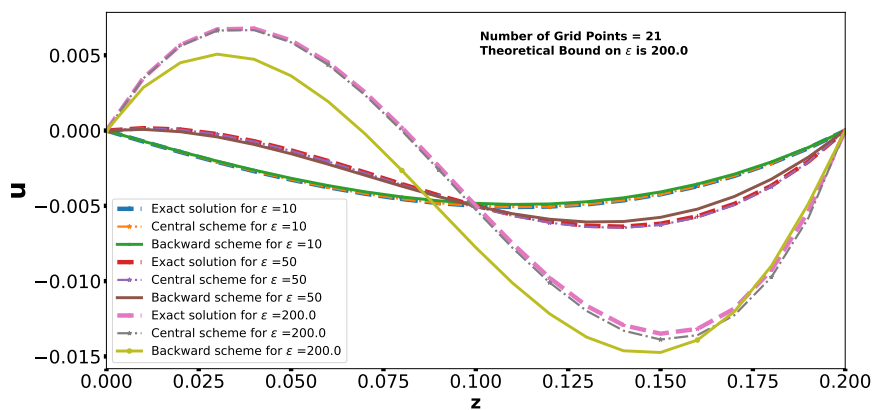


Fig. 4. Results with  $\epsilon = 10, 50, 200$  using 21 grid points

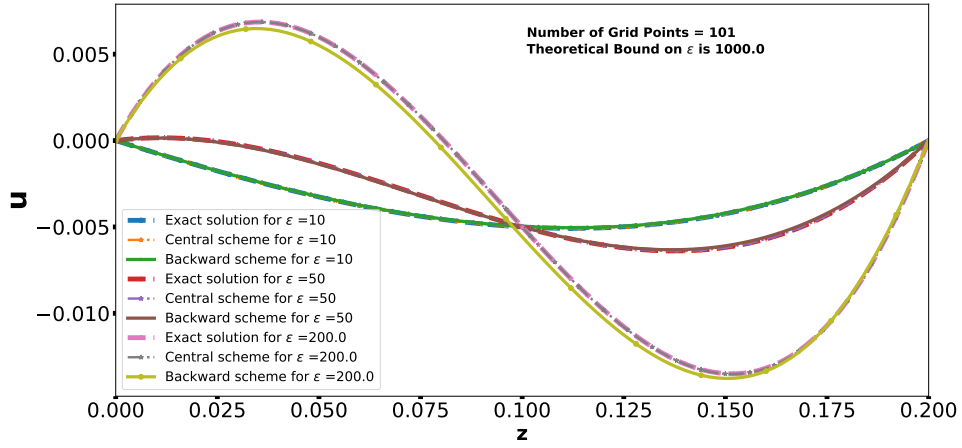


Fig. 5. Results with  $\epsilon = 10, 50, 200$  using 101 grid points

Finally, we take  $\epsilon$  to be very large values (up to millions). Figure 5 shows the results for  $1.0 \times 10^6 \leq \epsilon \leq 1.0 \times 10^7$  using 101 grid points, so that the theoretical bound on  $\epsilon$  for this grid is still 1000. We can see that both numerical schemes remain stable for values of  $\epsilon$  as large as one million. For  $\epsilon = 4.0 \times 10^6$  we can see that the central scheme becomes unstable (see the kicks), while the backward scheme becomes less accurate than it is for  $\epsilon = 1.0 \times 10^6$ . This is a numerical verification that the theoretical result truly proposes a safe bound for the stability of the central scheme. The instability becomes worse on increasing the parameter to ten million, see figure 6.

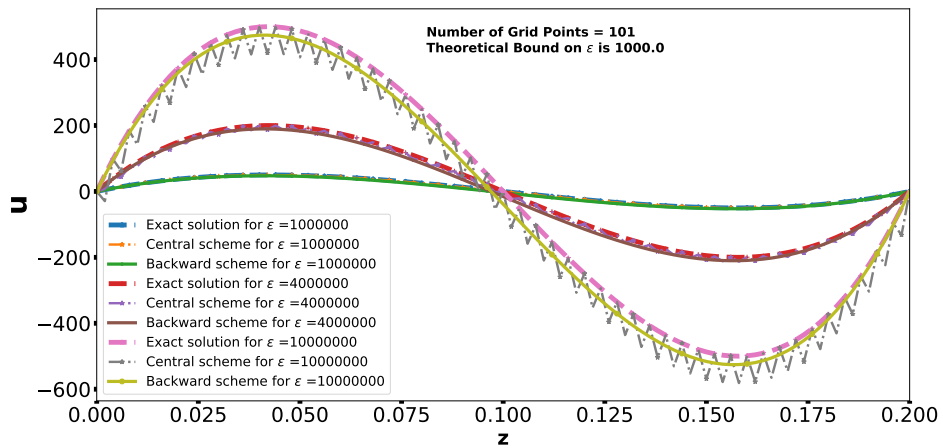


Fig. 6. Results with  $\epsilon = 1 \times 10^6, 4 \times 10^6, 1 \times 10^7$  using 101 grid points

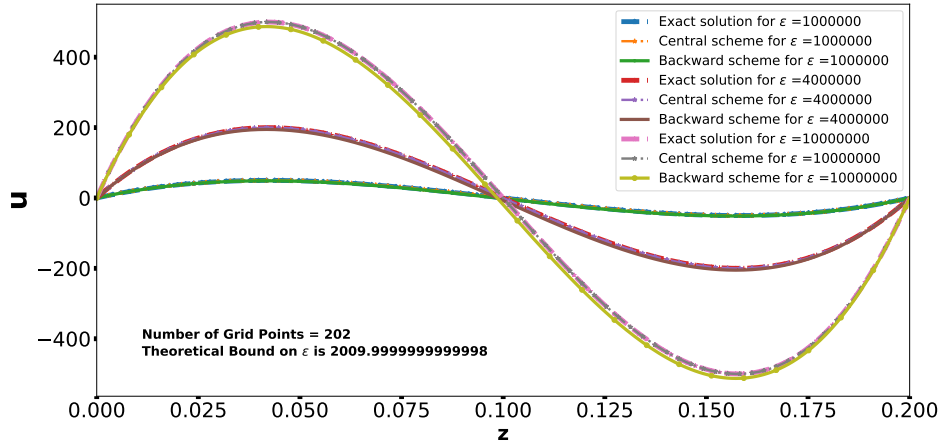


Fig. 7. Results with  $\epsilon = 1 \times 10^6, 4 \times 10^6, 1 \times 10^7$  using 202 grid points

Since the theoretical results relate the bound on the parameter to the grid spacing, it means that in order to eliminate the oscillatory behavior of the solutions the grid spacing has to be reduced. Hence, we decrease the grid spacing by using 202 grid points, the results are shown in figure 7. We can see that the oscillation disappears and the central scheme becomes highly accurate even more than the backward scheme.

## 5 Conclusion

Theoretical and numerical investigation of two numerical schemes have been carried out in this study. The finite difference methods are adopted because of their ease of use and implementation, moreover unlike the analytical methods, the scheme and code can be easily reused for more complicated problems without spending days doing hand calculations. One central and one backward scheme are considered and the influence of a model parameter analysed. It is theoretically proven and numerically verified that the stability of the central scheme is influenced by the value of a model parameter. We derive an estimate for the upper bound of the parameter. No numerical oscillation is observed below this theoretically obtained threshold. It is also observed that the theoretical threshold is a severe under-estimation of the numerically observed bound, hence we conclude that the theoretical bound is a very safe value to guarantee convergence.

## Acknowledgment

We are grateful to the anonymous reviewers for their valuable suggestions that improved the work. The first author is also appreciative of the Petroleum Technology Development Fund (PTDF) for sponsoring his PhD at Warwick during 2012-2016.

## Competing Interests

Authors have declared that no competing interests exist.

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