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Extensions of Locally Compact Abelian, Torsion-Free Groups by Compact Torsion Abelian Groups

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Authors' contributions

This work was carried out in collaboration between borh authors. Author HS designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript and managed literature searches. Author AAA managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.

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Abstract

Let X be a compact torsion abelian group. In this paper, we show that an extension of F_p by X splits where F_p is the p-adic number group and p a prime number. Also, we show that an extension of a torsion-free, non-divisible LCA group by X is not split.

Keywords: Locally compact abelian group; extension; divisible group; torsion-free group.

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1 Introduction

Throughout, all groups are Hausdorff abelian topological groups and will be written additively. Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called proper if it is open onto its image and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be proper exact if ϕ and ψ are proper morphisms. In this case the sequence is called an extension of A by C (in $\pounds$). Following [1], we let $Ext(C, A)$ denote the (discrete) group of extensions of A by C. The splitting problem in LCA groups is finding conditions on A and C under which $Ext(C, A) = 0$. In [2].[3],[4] the splitting problem is studied. We have studied the splitting problem in the category of divisible, LCA groups [5]. By using the splitting problem, we determined the LCA groups G such that the maximal torsion subgroup of G is closed [6]. Let X be a compact torsion group. In Theorem 1 of [3] , it is proved that if G is a divisible LCA group, then $Ext(X, G) = 0$. However, the suggested proof in [3] appears to be incomplete as it uses the incorrect Proposition 8 of [2]. In [5] , we proved that if G is a divisible, $\sigma-$compact group, then $Ext(X, G) = 0$. Let P be the set of all prime numbers, J_p, the p-adic integer group and F_p, the p-adic number group which is the minimal divisible extension of J_p for every $p \in P$ [7] . In this paper, we show that $Ext(X, F_p) = 0$ (see Lemma 2.2). By[7, 25.23] , a divisible, torsion-free LCA group G has the form $G \cong \mathbb{R}^n \bigoplus A \bigoplus M \bigoplus E$, where A is a discrete, torsion-free, divisible group, M a compact, connected, torsion-free group and E, the minimal divisible extension of $\prod_{p \in P} J_p^{n_p}$ where n_p is a cardinal number for every $p \in P$. Is $Ext(X, G) = 0$? We can not respond to this question in general. since, we do not know the structure of E. But, if I be a finite subset of P and n_p is finite for every $p \in I$, then $E = \prod_{p \in I} F_p^{n_p}$. In this paper, we show that if $G \cong \mathbb{R}^n \bigoplus A \bigoplus M \bigoplus \prod_{p \in I} F_p^{n_p}$, then $Ext(X, G) = 0$ (see Theorem 2.5).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals with discrete topology and $\mathbb{Z}$ is the group of integers. If $\{G_i\}_{i \in I}$ is a family of groups in $\pounds$, then we denote their direct product by $\prod_{i \in I} G_i$. If all the G_i are equal, we will write G^I instead of $\prod_{i \in I} G_i$. For any group G and H, $Hom(G, H)$ is the group of all continuous homomorphisms from G to H, endowed with the compact-open topology. The Pontryagin dual group of G is $\hat{G} = Hom(G, \mathbb{R}/\mathbb{Z})$. The topological isomorphism will be denote by " $\cong$ ". For more on locally compact abelian groups see [7].

2 Main Results

Lemma 2.1. *Let $X \in \pounds$ and p a prime number. Then $nExt(X, F_p) = Ext(X, F_p)$ for every positive integer n.*

Proof. Let n be a positive integer and $f : F_p \to F_p$, $f(x) = nx$ for all $x \in F_p$. By Lemma 2 of [8], f is open. So f is a proper morphism. Consider the exact sequence $0 \to Kerf \to F_p \xrightarrow{f} F_p \to 0$. By Corollary 2.10 of [1] , we have the exact sequence

$$\to Ext(X, Kerf) \to Ext(X, F_p) \xrightarrow{f_*} Ext(X, F_p) \to 0 \tag{2.1}$$

Since $f_*(Ext(X, F_p)) = nExt(X, F_p)$, it follows from sequence (2.1) that $nExt(X, F_p) = Ext(X, F_p)$. ☐

Lemma 2.2. *Let X be a compact torsion group. Then $Ext(X, F_p) = 0$.*

Proof. F_p is a totally disconnected group. So, by Theorem 24.30 of [7], F_p contains a compact open subgroup K. Now we have the following exact sequence

$$... \to Ext(X, K) \to Ext(X, F_p) \to Ext(X, F_p/K) \to 0 \qquad (2.2)$$

Since F_p is divisible, so $Ext(X, F_p/K) = 0$ (see Theorem 3.4 of [1]). Since X is compact and torsion, so by Theorem 25.9 of [7], $nX = 0$ for some positive integer n. Hence, $nExt(X, K) = 0$ (see Lemma 2.5 of [9] . Since (2.2) is exact, so $nExt(X, F_p) = 0$. Hence by Lemma 2.1, $Ext(X, F_p) = 0$. □

Remark 2.1. Let X be a group and $f : X \to X, f(x) = nx$ for all $x \in X$. If f is a topological isomorphism for every positive integer n, then X is a divisible, torsion-free group.

Theorem 2.3. *Let X be a compact group and p a prime number. Then $Ext(X, F_p)$ is a divisible, torsion-free group.*

Proof. Let n be a positive integer. Then the exact sequence $0 \to X \stackrel{\times n}{\to} X \to X/nX \to 0$ induces the following exact sequence

$$Ext(X/nX, F_p) \to Ext(X, F_p) \stackrel{\times n}{\to} Ext(X, F_p) \to 0$$

By Lemma 2.2, $Ext(X/nX, F_p) = 0$. So $Ext(X, F_p) \stackrel{\times n}{\to} Ext(X, F_p)$ is a topological isomorphism. Hence by Remark 2.1, $Ext(X, F_p)$ is a divisible, torsion-free group. □

Corollary 2.4. *Let $X \in £$. Then $Ext(X, F_p)$ is a divisible, torsion-free group.*

Proof. Let $X \in £$. By Theorem 24.30 of [7] , $X = \mathbb{R}^n \bigoplus H$ where H contains a compact open subgroup K. Consider the exact sequence

$$Ext(H/K, F_p) \to Ext(H, F_p) \to Ext(K, F_p) \to 0$$

Since H/K is a discrete group and F_p a divisible group, so $Ext(H/K, F_p) = 0$. Hence $Ext(H, F_p) \cong Ext(K, F_p)$. By Theorem 2.3, $Ext(K, F_p)$ is a divisible, torsion-free group. So $Ext(X, F_p)$ is a divisible, torsion-free group. □

Theorem 2.5. *Let X be a compact torsion group and $G \cong \mathbb{R}^n \bigoplus A \bigoplus M \bigoplus \prod_{p\in I} F_p^{n_p}$. Then $Ext(X, G) = 0$.*

Proof. First recall that by Theorem 2.13 of [1],

$$Ext(X, G) \cong Ext(X, A) \bigoplus Ext(X, M) \bigoplus \prod_{p\in I} Ext(X, F_p)$$

Since X is a totally disconnected group, so by Theorem 3.4 of [1] , $Ext(X, A) = 0$. Also $Ext(X, M) \cong Ext(\hat{M}, \hat{X})$ by Theorem 2.12 of [1] . Since $\hat{X}$ is a discrete bounded group and $\hat{M}$ a discrete torsion-free group, so by Theorem 27.5 of [10],$Ext(\hat{M}, \hat{X}) = 0$. By Lemma 2.2, $Ext(X, F_p) = 0$. Hence $Ext(X, G) = 0$. □

Lemma 2.6. *Let X be a compact torsion group. Then $Hom(X, \mathbb{Q}/\mathbb{Z}) \cong \hat{X}$.*

Proof. The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces the following exact sequence

$$Hom(X, \mathbb{Q}) \to Hom(X, \mathbb{Q}/\mathbb{Z}) \to Ext(X, \mathbb{Z}) \to Ext(X, \mathbb{Q})$$

Since X is torsion and $\mathbb{Q}$ is torsion-free, so $Hom(X, \mathbb{Q}) = 0$. Also by Theorem 3.4 of [1], $Ext(X, \mathbb{Q}) = 0$. Hence $Hom(X, \mathbb{Q}/\mathbb{Z}) \cong Ext(X, \mathbb{Z})$. By Theorem 2.12 & Proposition 2.17 of [1] , $Ext(X, \mathbb{Z}) \cong Ext(\hat{\mathbb{Z}}, \hat{X}) \cong \hat{X}$. So $Hom(X, \mathbb{Q}/\mathbb{Z}) \cong \hat{X}$. □

Theorem 2.7. *Let X be a compact torsion group and G a torsion-free, non-divisible group. Then $Ext(X, G) \neq 0$.*

Proof. Let G^* be the minimal divisible extension of G. By A.13 of[7], G^* is a divisible, torsion-free group. Since X is torsion and G^* torsion-free, so $Hom(X, G^*) = 0$. By Corollary 2.10 of [1] , we have the following exact sequence

$$0 = Hom(X, G^*) \to Hom(X, G^*/G) \to Ext(X, G)$$

Since G^*/G is a discrete, torsion divisible group, so $Hom(X, G^*/G)$ containing a copy of $Hom(X, \mathbb{Q}/\mathbb{Z})$. Hence by Lemma 2.6, $Ext(X, G) \neq 0$. □

Corollary 2.8. *Let X be a compact torsion group and G a torsion-free group. If $Ext(X, G) = 0$, then G is a divisible group.*

3 Conclusion

Let X be a compact torsion abelian group.In this paper, we show that an extension of a torsion-free, non-divisible LCA group by X is not split.

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Competing Interests

Authors have declared that no competing interests exist.

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