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The Boundedness of High Order Riesz-Bessel Transforms via Atomic-molecular Characterization on Weighted $H^p_{\triangle_\nu,\omega}$ Spaces

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Authors' contributions

This work was carried out in collaboration between the two authors. Author IE presented the problem, designed the study, performed the analysis and managed the literature search. Author CK solved the problems and wrote the manuscript. Both authors read and approved the final manuscript.

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Abstract

Let $0 < p \leq 1$ and w in the Muckenhoupt class $A_{p,\nu}$. In this paper, we will characterize the weighted Hardy spaces in the setting of the Laplace-Bessel differential operators by means of the high order Riesz-Bessel transforms generated by generalized shift operator. We give the elemantary result that for certain weight functions w. We also obtain the $H^p_{\triangle_\nu,\omega}$ boundedness of high order Riesz-Bessel transform within the bounds of some necessary conditions through the atomic-molecular characterization.

Keywords: Atomic decomposition; $A_{p,\nu}$ weights; generalized shift operator; Hardy space; Riesz-Bessel transform.

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1 Introduction

The theory of Hardy spaces H^p has its origin in the extraordinary discoveries made eighty years ago. It is one of the most popular work in harmonic analysis and has many applications in partial differential equations. However, there are important situations in which the standard theory of Hardy spaces is not applicable, including certain problems in the theory of partial differential equation which involves generalizing the Laplacian. There is a need to consider Hardy spaces that are adapted to a differential operator Laplace-Bessel, similarly to the way that the standard theory of Hardy spaces are adapted to the Laplacian.

It is well known that the Riesz transforms are bounded on classical Hardy spaces. There are many different approaches to show this classical result [1, 2]. Classical weighted Hardy spaces H^p_w have been extensively by [3] and [4], here w is Muckenhoupt's A_p weight. Also, the atomic characterization of this space has been given by these two authors. Recently, by using the weighted atom-molecule theory and combined with Garcia-Cuerva's atomic decomposition for weighted Hardy spaces $H^p_w(\mathbb{R}^n)$, the authors in [2] established that the classical Riesz transforms $R_j,\ j = 1, \ldots, n$ are bounded on $H^p_w(\mathbb{R}^n)$. However, in [5], authors defined molecules for weighted Hardy spaces and proved their molecule characters. As an application they gave sufficient conditions on the kernel k such that the convolution operator $Tf = k * f$ is bounded on weighted Hardy spaces $H^p_w(\mathbb{R}^n)$, $w \in A_1$. They also get the $H^p_w(\mathbb{R}^n)$, $\frac{1}{2} < p \leq 1$, boundedness of the Hilbert transform and the $H^p_w(\mathbb{R}^n)$, $\frac{n}{n+1} < p \leq 1$, boundedness of the Riesz transforms.

The goal of this paper is to study the weighted version of the $H^p_{\triangle_\nu,\omega}$ Hardy space, where w is assumed to satisfy the condition $A_{p,\nu}$ of Muckenhoupt. We identify the high order Riesz-Bessel transforms (3.1) associated to the Laplace-Bessel differential operator, and also show that this singular integral operator are bounded on $H^p_{\triangle_\nu,\omega}$ Hardy space setting by atomic-molecular characterization.

Throughout this paper, C denotes a positive constant which is independent of the parameters, but may change from line to line. Also, a subscript is added when we want to make clear its dependence on the parameter in the subscript.

Let us briefly describe our motivations. Denote $\mathbb{R}^n_+$, the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$, defined by the inequality $x_n > 0$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $B(x, r) = \{y \in \mathbb{R}^n_+ \ ; \ |x - y| < r\}$. For any measurable set $B \subset \mathbb{R}^n_+$ we define $|B|_\nu = \int_B x_n^\nu dx$, where $\nu > 0$. Then $|B(0, r)|_\nu = \omega(n, \nu) r^Q, Q = n + \nu$, where

$$\omega(n,\nu) = \int_{B(0,1)} x_n^\nu dx = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{Q-2}{2}\right)}.$$

An almost everywhere positive and locally integrable function $w : \mathbb{R}^n_+ \to \mathbb{R}$ will be called a weight. We shall denote by $L^p_{w,\nu} = L^p_{w,\nu}(\mathbb{R}^n_+)$, $1 \leq p < \infty$ the space of all measurable functions f with a finite norm

$$\|f\|_{L^p_{w,\nu}} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p w(x) x_n^\nu dx \right)^{1/p}.$$

For the case $w = 1$, the space $L^p_{w,\nu}(\mathbb{R}^n_+)$ is denoted by $L^p_\nu(\mathbb{R}^n_+)$, and the norm $\|f\|_{L^p_{w,\nu}}$ by $\|f\|_{L^p_\nu}$.

The Fourier-Bessel transform on $\mathcal{S}_+$, Schwartz space of rapidly decreasing smooth functions, are defined by

$$F_\nu f(x) = \int_{\mathbb{R}^n_+} f(y)\, e^{-i(x'y')} j_{\frac{\nu-1}{2}}(x_n y_n)\ y_n^\nu dy, \tag{1.1}$$

where $(x'y') = x_1y_1 + \ldots + x_{n-1}y_{n-1}$, j_ν, $\nu > 0$, is the normalized Bessel function, and $C_{n,\nu} = (2\pi)^{n-1}2^{\nu-1}\Gamma^2((\nu+1)/2) = \frac{2}{\pi}\omega(2,\nu)$.

This transform is associated to the Laplace-Bessel differential operator

$$\Delta_\nu = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n}, \quad \nu > 0.$$

The expression (1.1) is a hybrid of the Hankel transform in the x_n-variable and the ordinary Fourier transform on $\mathbb{R}^{n-1}$. These transforms and related problems for singular PDE and fractional integrals were studied by I. A. Kipriyanov and his collaborators K. Trimeche, by J. Peetre, K. Stempak, I. A. Aliev, V. S. Guliyev, I. Ekincioglu, and others.

The generalized shift operator is defined by the following way:

$$T^y f(x) = C_\nu \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_ny_n\cos\theta + y_n^2}\right)\sin^{\nu-1}\theta d\theta,$$

where $C_\nu = \pi^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)[\Gamma\left(\frac{\nu}{2}\right)]^{-1}$ (see [6, 7]). Note that this operator is closely connected with Laplace-Bessel differential operator.

Let $f, g \in L_\nu^p(\mathbb{R}_+^n)$ and T^y be the generalized shift operator. Then, the expression

$$(f \otimes g)(x) = \int_{\mathbb{R}_+^n} f(y)T^y g(x) y_n^\nu dy$$

is called Δ_ν-convolution.

2 $A_{p,\nu}$ Weights, Atoms and Molecules

In this paper, a weight means the $A_{p,\nu}$ weight. Let us recall the definition and properties of $A_{p,\nu}$.

Definition 2.1. The weight function $w \in A_{p,\nu}(\mathbb{R}_+^n)$ for $1 < p < \infty$, if

$$\sup_{x\in\mathbb{R}_+^n} |E|_\nu^{-1} \int_E w(y)y_n^\nu dy \left(|E|_\nu^{-1}\int_E w^{\frac{-1}{p-1}}(y)y_n^\nu dy\right)^{p-1} < \infty$$

and $w \in A_{1,\nu}(\mathbb{R}_+^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}_+^n$

$$|E|_\nu^{-1}\int_E w^{\frac{-1}{p-1}}(y)y_n^\nu dy \le C \operatorname{ess}\inf_{y\in E} w(y).$$

$A_{\infty,\nu} = \cup_{p>1}A_{p,\nu}$. The properties of the class $A_{p,\nu}(\mathbb{R}_+^n)$ are analogous to those of B. Muckenhoupt classes. In particular, if $w \in A_{p,\nu}(\mathbb{R}_+^n)$ for $1 < p < \infty$, then $w \in A_{r,\nu}(\mathbb{R}_+^n)$ for all $r > p$ and $w \in A_{q,\nu}(\mathbb{R}_+^n)$ for any $1 < q < p$. We denote by $q_{w,\nu} = \inf\{q > 1 : w \in A_{q,\nu}\}$ the critical index of w, and set the weighted measure $w(E) = |E|_{\nu,w} = \int_E w(x)x_n^\nu dx$. It is clear that this measure satisfies the following doubling condition [8, 9].

Lemma 2.1. *Let $w \in A_{p,\nu}$, $p \ge 1$. Then, for any cube E and and $\lambda > 1$,*

$$w(\lambda E) \le C\lambda^{(n+\nu)p}w(E),$$

where C is independent of cube E and fixed point λ.

A close relation to $A_{p,\nu}$ is the reverse Hölder inequality. If there exist $r > 1$ and a positive constant $C > 0$ such that

$$\left(\frac{1}{|E|_\nu}\int\limits_E w^r(x)x_n^\nu dx\right)^{\frac{1}{r}} \leq C\left(\frac{1}{|E|_\nu}\int\limits_E w(x)x_n^\nu dx\right),$$

for every cube $E \subset \mathbb{R}_+^n$, we say that w satisfies the reverse Hölder condition of order r and write $w \in RH_{r,\nu}$. It follows from this inequality that $w \in RH_{r,\nu}$ implies $w \in RH_{s,\nu}$ for $s < r$.

Lemma 2.2. *Let $w \in A_{q,\nu} \cap RH_{r,\nu}$, $q \geq 1$ and $r > 1$. Then there exists constants $C_1, C_2 > 0$ such that*

$$C_1\left(\frac{|I|}{|E|}\right)^q \leq \frac{w(I)}{w(E)} \leq C_2\left(\frac{|I|}{|E|}\right)^{\frac{r-1}{r}},$$

for any measurable subset I of a cube E.

Lemma 2.3. *Let $r > 1$. Then $w^r \in A_{\infty,\nu}$ if and only if $w \in RH_{r,\nu}$.*

These two lemmas above are easily obtained using by similar method in [10].

Similarly to the classical Hardy spaces, the weighted Hardy space $H^p_{\triangle_\nu,w}$, $p > 1$ can be defined in terms of maximal functions. Let φ be a function in $\mathcal{S}(\mathbb{R}_+^n)$, Schwartz space of rapidly decreasing smooth functions with $\int_{\mathbb{R}_+^n}\varphi(x)\ x_n^\nu dx = 1$. Define $\varphi_t(x) = t^{-n-\nu}\varphi(t^{-1}x), t > 0, x \in \mathbb{R}_+^n$, and the B-maximal function $\mathcal{M}_\nu f$ by $\mathcal{M}_\nu f(x) = \sup\limits_{t>0}|(f \otimes \varphi_t)(x)|$. Then $H^p_{\triangle_\nu,w}$ consists of those tempered distributions $f \in \mathcal{S}^{'}(\mathbb{R}_+^n)$ for which $\mathcal{M}_\nu f \in L^p_{w,\nu}(\mathbb{R}_+^n)$ with $\|f\|_{H^p_{\triangle_\nu,w}} = \|\mathcal{M}_\nu f\|_{L^p_{\nu,w}}$. In the theory of Hardy spaces, the atomic decomposition theory is very important. In the weighted Hardy spaces, there is also atomic decomposition theory.

The notion of atoms and the so-called atomic decomposition theorem in $H^p_{\Delta_\nu}$ space is as follows:

Definition 2.2. Let $0 < p \leq 1 \leq q \leq \infty$ with $p \neq q$ such that $w \in A_{q,\nu}$ with critical index $q_{w,\nu}$. Set [.] as the integer function. A (p, q, s)-atom $a(x)$ is a function in $L^q_{w,\nu}(\mathbb{R}_+^n)$ which satisfies the following properties:

(i) a be supported on a cube E, namely, supp $a \subset E$,

(ii) $||a(x)||_{L^q_{w,\nu}} \leq w(E)^{\frac{1}{q}-\frac{1}{p}}$,

(iii) $\int_E a(x)x^\alpha x_n^\nu dx = 0$ for all s with $|\alpha| \leq s,\ s \geq [(Q+k)\left(\frac{q_{w,\nu}}{p} - 1\right)]$.

Theorem 2.4. *Suppose $0 < p \leq 1$ and $w \in A_{\infty,\nu}$. Then for any $f \in H^p_{\triangle_\nu,w}$, there exist a sequence of scalars λ_j, and a sequence $a_j = \{a_j\}$ of (p, ∞, s)-atoms with respect to w such that $f = \sum\limits_{j=1}^\infty \lambda_j a_j$ with*

$$\sum_{j=1}^\infty |\lambda_j|^p \leq C||f||^p_{H^p_{\triangle_\nu,w}}.$$

Let $H^{p,q,s}_{\triangle_\nu,w}$ denote the space consisting of all tempered distributions ,for which belong to $L^p_{w,\nu}(\mathbb{R}_+^n)$ weighted Lebesgue spaces, admitting a decomposition $f = \sum\limits_{j=1}^\infty \lambda_j a_j$, where a_j's are w-(p, q, s) atoms. If $\{a_j\}$ is a sequence of $H^p_{\Delta_\nu,w}$-atoms and $\{\lambda_j\} \in \ell_p$, then the series

$$f = \sum_{j=1}^\infty \lambda_j a_j$$

converges in the sense of distributions and its sum f belongs to $H^p_{\Delta_\nu,w}$ with

$$\mathcal{N}_{p,q,s}(f) = \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{\frac{1}{p}}\right\},$$

where the infimum is taken over all the decompositions of f.

Moreover, we have the following properties.

Theorem 2.5. *If both triples (p, q_1, s) and (p, q_2, s) satisfy the conditions in Definition 2.2, then $H^{p,q_1,s}_{\Delta_\nu,w} = H^{p,q_2,s}_{\Delta_\nu,w}$. For all q, the sizes $\mathcal{N}_{p,q,s}(f)$ are equivalent.*

Theorem 2.6. *All (p, q, s)-atoms with respect to weight w are elements of $H^p_{\Delta_\nu,w}$ with its $H^p_{\Delta_\nu,w}$ norm is bounded by a constant independent of the atom.*

Theorem 2.7. *All spaces $H^{p,q,s}_{\Delta_\nu,w}$ coincide with $H^p_{\Delta_\nu,w}$ and $\mathcal{N}_{p,q,s}(f) \approx ||f||_{H^p_{\Delta_\nu,w}}$ provided the triple (p, q, s) satisfies the conditions in Definition 2.2.*

The weighted molecules corresponding to the weighted atoms described above can be defined in the following.

Definition 2.3. For $0 < p \le 1 \le q \le \infty$ and $p \ne q$, let $w \in A_{q,\nu}$ with critical index $q_{w,\nu}$ and critical index $r_{w,\nu}$ for the reverse Hölder condition. Set $s \ge [(Q + k)\big(\frac{q_{w,\nu}}{p} - 1\big)]$, $\epsilon > \max\{sr_{w,\nu}(r_{w,\nu} - 1)^{-1}Q^{-1} + (r_{w,\nu} - 1)^{-1}, 1/p - 1\}$, $a = 1 - \frac{1}{p} + \epsilon$ and $b = \frac{1}{2} + \epsilon$. A (p, q, s, ϵ)-molecule with respect to weight w is a function $M(x) \in L^q_{w,\nu}(\mathbb{R}^n_+)$ which satisfies the following properties:

(i) $M(x).w(E)^b \in L^q_{w,\nu}(\mathbb{R}^n_+)$,

(ii) $||M||^{a/b}_{L^q_{w,\nu}}.||M(x).w(E)^b||^{1-a/b}_{L^q_{w,\nu}} := \mathfrak{N}_{w,\nu}(M) < \infty$, (molecular norm of M with respect to w)

(iii) $\int_{\mathbb{R}^n_+} M(x)x^\alpha x_n^\nu dx = 0$ for all s with $|\alpha| \le s$.

For the boundedness of operators on $H^p_{\Delta_\nu,w}$ spaces, the most interesting problem is to find what kind of operators are bounded on this space. An application of atom-molecule theory, this problem is easily reduced to whether the image of weighted (p, q, s)-atom under the action of the operator is a weighted (p, q, s, ϵ)-molecule.

Before we state our main results, we need the following lemma.

Lemma 2.8. *Let $w \in A_{q,\nu}$, $q > 1$. Then, for all $r > 0$, there exists a constant C independent of R such that*

$$\int\limits_{|x|\ge r} \frac{w(x)}{|x|^{(Q+k)q}} x_n^\nu dx \le C r^{-(Q+k)q} w(I_r)$$

where I_r the cube centered at origin with side length $2r$.

3 Boundedness of Weighted Hardy Spaces $H^p_{\Delta_\nu,w}$

The main purpose of what is showed in this section is that, based on the atomic decomposition and molecular characterization method, to investigate the action of singular integrals especially the high order Riesz-Bessel transforms on weighted $H^p_{\Delta_\nu,w}$ Hardy spaces.

Now we consider the high order Riesz-Bessel transforms. Let $1 \leq p < \infty$, $f \in L^p_{w,\nu}$ and w be a weight function on $\mathbb{R}^n_+$. The high order Riesz-Bessel transform of f is defined by setting,

$$\begin{aligned} R^{(k)}_\nu(f)(x) &= C_{k,\nu}\big[\ p.v\ \big(\tfrac{P_k(y)}{|y|^{Q+k}} \otimes f\big)\big](x) \\ &= C_{k,\nu} \lim_{\varepsilon \to 0} \int_{\varepsilon < |y|} \frac{P_k(y)}{|y|^{Q+k}} T^y f(x) y_n^\nu dy, \end{aligned} \tag{3.1}$$

where $C_{k,\nu} = 2^{\frac{n+\nu}{2}} \Gamma(\frac{n+k+\nu}{2}) \big[\Gamma(\frac{k}{2})\big]^{-1}$ and $P_k(y) = P_k(y_1, y_2, \ldots, y_{n-1}, y_n^2)$ is a homogeneous polynomial of degree k which holds $\triangle_\nu P_k(y) = 0$ on $\mathbb{R}^n_+$. Moreover, this polynomial satisfies the cancellation condition

$$\int_{S_+} P_k(\theta)(\theta')^\nu d\theta = 0 \tag{3.2}$$

and

$$\sup_{\theta \in S_+} |P_k(\theta)| = A < \infty \tag{3.3}$$

where $S_+ = \{y \in \mathbb{R}^n_+ : |y| = 1\}$ and $\theta = \frac{y}{|y|}$.

We also have the following $L^p_{w,\nu}$ boundedness of high order Riesz-Bessel transform.

Theorem 3.1. *Suppose that the characteristic P_k of the singular integral (3.1) satisfies the conditions (3.2) and (3.3). Moreover, let $1 < p < \infty$ and $w(x)$ be weight function on $\mathbb{R}^n_+$. Then there exists a constant $C > 0$, independent of f such that for all $f \in L^p_{w,\nu}$*

$$\int_{\mathbb{R}^n_+} |R^{(k)}_\nu(f)(x)|^p w(x) x_n^\nu dx \leq C \int_{\mathbb{R}^n_+} |f(x)|^p w(x) x_n^\nu dx$$

(see [11]).

Let $K(x)$ be a function defined as $R^{(k)}_\nu f := (K \otimes f)$. This kernel satisfies the smoothness estimate,

$$\int_{|x| \geq 2|y|} |T^y K(x) - K(x)| x_n^\nu dx \leq C,$$

for some $C > 0$ (See in detail [12]). It is a very useful fact that this properties and the L^2_ν-boundedness of high order Riesz-Bessel transforms, to be getting of maps $H^p_{\Delta_\nu}$ to L^p_ν and $H^p_{\Delta_\nu}$ to $H^p_{\Delta_\nu}$ for $0 < p \leq 1$.

We now are able to extend the results to the weighted case.

Theorem 3.2. *Let $R^{(k)}_\nu(f)$ be a singular integral operator related to generalized shift operator, and $w \in A_{p,\nu}$ for $1 < p < \infty$. Then this operator can be extended to be a bounded operator for $L^p_{w,\nu}$ to itself.*

We continue with $(H^1_{\triangle_\nu,w}, L^1_{w,\nu})$ boundedness of high order Riesz-Bessel transforms. It is known that high order Riesz-Bessel transforms are bounded in $L^p_{w,\nu}$ spaces for all $1 < p < \infty$. Therefore, this operator is well defined on $L^p_{w,\nu}$ and in particular on $H^p_{\triangle_\nu,w}$ which is contained in $L^p_{w,\nu}$. In the case $p = 1$, the following theorem holds.

Theorem 3.3. *Let $R^{(k)}_\nu f := (K \otimes f)$ be defined as in (3.1) and $w \in A_{1,\nu}$. Assume that K satisfies*

$$\int_{|x| \geq 2|y|} |T^y K(x) - K(x)| w(x) x_n^\nu dx \leq C_1 w(y), \tag{3.4}$$

and

$$||K \otimes f||_{L^2_{w,\nu}} \leq C_2||f||_{L^2_{w,\nu}}. \tag{3.5}$$

Then there is a constant C independent of f such that for all $f \in H^1_{\Delta_\nu,w}(\mathbb{R}^n_+)$ we have

$$||K \otimes f||_{L^1_{w,\nu}} \leq C||f||_{H^1_{\Delta_\nu,w}}.$$

Proof. By the atomic decomposition of Hardy spaces, it suffices to show that $||(K \otimes f)||_{L^1_{w,\nu}} \leq C$ for any w-$(1,2,0)$-atom f with constant C independent of the choice of f. Given w-$(1,2,0)$-atom f with supp$(f) \subseteq I_R$, we have $||f||_{L^2_{w,\nu}} \leq w(I_R)^{\frac{-1}{2}}$ and $\int f(x)x_n^\nu dx = 0$. Hence, we can write

$$\begin{aligned} ||(K \otimes f)||_{L^1_{w,\nu}} &= \int_{\mathbb{R}^n_+} |(K \otimes f)(x)|w(x)x_n^\nu dx \\ &= \left(\int_{|x|\geq 2\sqrt{n}R} + \int_{|x|<2\sqrt{n}R}\right)|(K \otimes f)(x)|w(x)x_n^\nu dx = I + II. \end{aligned}$$

Let us consider the estimate I. Applying the Fubini's theorem and by the inequality (3.4), we have

$$\begin{aligned} I &= \int_{|x|\geq 2\sqrt{n}R} |(K \otimes f)(x)|w(x)x_n^\nu dx \\ &= \int_{|x|\geq 2\sqrt{n}R} \left|\int T^y K(x) f(y) y_n^\nu dy\right| w(x)x_n^\nu dx \\ &\leq \int_{I_R} |f(y)| \left\{\int_{|x|\geq 2|y|} |T^y K(x) - K(x)|w(x)x_n^\nu dx\right\} y_n^\nu dy \\ &\leq C_1 \int_{I_R} |f(y)|w(y)y_n^\nu dy \leq C. \end{aligned} \tag{3.6}$$

On the other hand by Schwartz's inequality and the doubling condition, we obtain

$$\begin{aligned} II &= \int_{|x|<2\sqrt{n}R} |(K \otimes f)(x)|w(x)x_n^\nu dx \\ &\leq ||(K \otimes f)||_{L^2_{w,\nu}} \left(\int_{|x|<2\sqrt{n}R} w(x)x_n^\nu dx\right)^{\frac{1}{2}} \leq C_2||f||_{L^2_{w,\nu}} w(2\sqrt{n}I_R)^{\frac{1}{2}} \leq C. \end{aligned} \tag{3.7}$$

Both inequalities (3.6) and (3.7) get $||(K \otimes f)||_{L^1_{w,\nu}} \leq C$ for any w-$(1,2,0)$ atom f, and this finishes the proof. ☐

Here we will show similar results held for $H^p_{\triangle_\nu}$ Hardy spaces for $p < 1$ providing that strengthen the assumption on the kernel K.

Theorem 3.4. *Let $w \in A_{1,\nu}$ with critical index $r_{w,\nu}$ for the reverse Hölder condition. Suppose that $K \in L_{loc,\nu}(\mathbb{R}^n_+)$ satisfies the $L^2_{w,\nu}$-boundedness (3.5) and holds the following inequality*

$$\left|T^y K(x) - K(x)\right| \leq C\frac{|y|^\lambda}{|x|^{Q+k+\lambda}} \quad \textit{whenever} \quad |x| \geq 2|y| \tag{3.8}$$

for some $0 < \lambda \leq 1$. If $r_{w,\nu} > (Q+k+\lambda)/\lambda$, then the operator $R^{(k)}_\nu f$ is bounded on $H^p_{\Delta_\nu,w}(\mathbb{R}^n_+)$, $(Q+k)/(Q+k+\lambda) < p \leq 1$.

Proof. Let $(Q+k)/(Q+k+\lambda) < p \leq 1$ and $r_{w,\nu} > (Q+k+\lambda)/\lambda$. It can be easily seen that $[(Q+k)(1/p-1)] = 0$ and $\max\{1/(r_{w,\nu}-1), 1/p-1\} < \lambda/(Q+k)$. We can choose ϵ satisfying $\max\{1/(r_{w,\nu}-1), 1/p-1\} < \epsilon < \lambda/(Q+k)$. We want to show that for every $w-(p,\infty,0)$-atom f centered at the origin, its Riesz-Bessel transform is a $w-(p,2,0,\epsilon)$-molecule. and $\mathfrak{N}_{w,\nu}(R^{(k)}_\nu f) \leq C$.

By the definition of atom, for any $w-(p,\infty,0)$-atom f with supp$(f) \subseteq I_R$, $||f||_{L^{\infty}_{w,\nu}} \leq w(I_R)^{\frac{-1}{p}}$ and $\int f(x)x_n^{\nu}dx = 0$. Let us denote $a = 1 - \frac{1}{p} + \epsilon$ and $b = \frac{1}{2} + \epsilon$. Then,

$$
\begin{aligned}
||R_{\nu}^{(k)} f(x) w(I_{|x|})^b||^2_{L^2_{w,\nu}} &= \int_{\mathbb{R}^n_+} |R_{\nu}^{(k)} f(x)|^2 w(I_{|x|})^{1+2\epsilon} w(x) x_n^{\nu} dx \\
&= \left(\int_{|x|<2\sqrt{n}R} + \int_{|x|\geq 2\sqrt{n}R} \right) |R_{\nu}^{(k)} f(x)|^2 w(I_{|x|})^{1+2\epsilon} w(x) x_n^{\nu} dx \\
&= I^{'} + II^{''}.
\end{aligned}
$$

The $L^2_{w,\nu}$-boundedness of $R_{\nu}^{(k)}$ yields

$$
\begin{aligned}
I^{'} &\leq C_{n,\nu,w} w(I_R)^{1+2\epsilon} ||R_{\nu}^{(k)} f||^2_{L^2_{w,\nu}} \\
&\leq C_{n,\nu,w} w(I_R)^{1+2\epsilon} ||f||^2_{L^2_{w,\nu}} \leq C_{n,\nu,w} w(I_R)^{2a}.
\end{aligned}
$$

To estimate part $II^{'}$, we obtain

$$
\begin{aligned}
II^{'} &= \int_{|x|\geq 2\sqrt{n}R} |(K \otimes f)(x)|^2 w(I_{|x|})^{1+2\epsilon} w(x) x_n^{\nu} dx \\
&= \int_{|x|\geq 2\sqrt{n}R} \left| \int_{I_R} \{T^y K(x) - K(x)\} f(y) y_n^{\nu} dy \right|^2 w(I_{|x|})^{1+2\epsilon} w(x) x_n^{\nu} dx.
\end{aligned}
$$

Let us investigate the inner integral. By the Schwartz's inequality and the condition (3.8) on kernel K, we have

$$
\begin{aligned}
&\left| \int_{I_R} \{T^y K(x) - K(x)\} f(y) y_n^{\nu} dy \right|^2 \leq \left\{ \int_{I_R} |T^y K(x) - K(x)| |f(y)| y_n^{\nu} dy \right\}^2 \\
&\leq C_{n,v} R^{2Q+2\lambda} |x|^{-2Q-2k-2\lambda} ||f||^2_{L^{\infty}_{w,\nu}}.
\end{aligned}
$$

Applying Lemma 2.2 and 2.8, we obtain

$$
\begin{aligned}
II^{'} &\leq C_{n,\nu} R^{2Q+2\lambda} ||f||^2_{L^{\infty}_{w,\nu}} \int_{|x|\geq 2\sqrt{n}R} |x|^{-2Q-2k-2\lambda} w(I_{|x|})^{1+2\epsilon} w(x) x_n^{\nu} dx \\
&\leq C_{n,w,\nu} R^{Q+2\lambda-2Q\epsilon} ||f||^2_{L^{\infty}_{w,\nu}} w(I_R)^{1+2\epsilon} \int_{|x|\geq 2\sqrt{n}R} |x|^{2Q\epsilon-Q-2k-2\lambda} w(x) x_n^{\nu} dx \\
&\leq C_{n,w,\nu,R} w(I_R)^{2a}.
\end{aligned}
$$

Thus,

$$
||R_{\nu}^{(k)} f(x) w(I_{|x|})^b||_{L^2_{w,\nu}} \leq C_{n,w,\nu,k} w(I_R)^a
$$

and

$$
\mathfrak{N}(R_{\nu}^{(k)} f) = ||Tf||^{\frac{a}{b}}_{L^2_{w,\nu}} ||R_{\nu}^{(k)} f(x) w(I_{|x|})^b||^{1-\frac{a}{b}}_{L^2_{w,\nu}} \leq C_{n,w,\nu,R}.
$$

The last property is easily obtained by Fourier-Bessel transform. The moment condition of f gives $F_{\nu}[f(0)] = 0$ which implies $F_{\nu}[R_{\nu}^{(k)} f](0) = F_{\nu}[K(0)] F_{\nu}[f(0)] = 0$. So we then have $\int R_{\nu}^{(k)} f(x) x_n^{\nu} dx = 0$ and this is the end of the proof. □

Theorem 3.5. *Let $w \in A_{1,\nu}$ with critical index $r_{w,\nu}$ and $s > (Q+k)/(r_{w,\nu}-1)$. Suppose that K be a function on $\mathbb{R}^n_+ \setminus \{0\}$ satisfies*

$$|D^\alpha_\nu T^y K(x)| \leq CA|x|^{-Q-k-|\alpha|}$$

for all multi-indices $|\alpha| \leq s$. Then the high order Riesz-Bessel transforms are bounded on $H^p_{\Delta_\nu,w}$ for $(Q+k)/(Q+k+1) < p \leq 1$.

Proof. Similarly to the arguments in Theorem 3.3, replacing (3.6) by

$$\begin{aligned} I &= \int\limits_{|x|\geq 2\sqrt{n}R} |(K\otimes f)(x)|^2 w(I_{|x|})^{1+2\epsilon} w(x) x_n^\nu dx \\ &= \int\limits_{|x|\geq 2\sqrt{n}R} \left| \int_{I_R} \left[T^y K(x) - \sum_{|\alpha|\leq s-1} D^\alpha_\nu T^y K(x) \tfrac{(-y)^\alpha}{\alpha!}\right] f(y) y_n^\nu dy \right|^2 w(I_{|x|})^{1+2\epsilon} w(x) x_n^\nu dx \end{aligned}$$

we obtain the estimate

$$\left| \int_{I_R} \left[T^y K(x) - \sum_{|\alpha|\leq s-1} D^\alpha_\nu T^y K(x) \tfrac{(-y)^\alpha}{\alpha!}\right] f(y) y_n^\nu dy \right|^2 \leq C_{n,\nu} R^{2Q+2s} |x|^{-2Q-2k-2s} ||f||^2_{L^\infty_{w,\nu}}$$

for $|x| \geq 2\sqrt{n}R$. If we follow the same way to prove this theorem as in Theorem 3.4, the proof is complete. □

4 Conclusions

On this paper, we wish to remark some points. In the section 1, a brief on the historical evolution and applications of weighted Hardy spaces has been discussed. In the section 2, we have described $A_{p,\nu}$ weights and its properties, atoms and molecules. We have given some lemmas and theorems about relations between these. In the final section we have defined the high order Riesz-Bessel transforms associated to the Laplace-Bessel differential operator, and also have shown that this singular integral operator are bounded on $H^p_{\triangle_\nu,\omega}$ Hardy space, $0 < p \leq 1$, via atomic-molecular characterization.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Lee MY. Different approaches to the H^p boundedness of Riesz transforms. Commun. Math. Anal. 2006;1(2):101-108.

[2] Lee MY, Lin CC, Yang WC. H^p_w-boundedness of Riesz transforms. J.Math. Anal. Appl. 2005;301:394-400.

[3] Garcia-Cuerva J. Weighted H^p spaces. Dissertations Math. 1979;162:1-63.

[4] Stromberg JO, Torchinsky A. Weighted hardy spaces. Lecture Notes in Mathematics. Springer-Verlag. 1989;1381.

[5] Lee MY, Lin CC. The molecular characterization of weighted Hardy spaces. J. Funct. Anal. 2002;188:442-460.

[6] Kipriyanov IA. Fourier-Bessel transformations and imbedding theorems. Tr.Math.Im.V.A.Steklova Akad. Nauk. SSSR. 1967;89:130-213.

[7] Levitan BM. Bessel function expansions in series and Fourier integrals. Uspekhi Mat. Nauk. 1951;6(2):102-143 (in Russian).

[8] Guliyev EV. Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator. Trans. of NAS of Azerbj. 2005:71-80.

[9] Guliyev EV. Weighted inequality for some sublinear operators in Lebesgue spaces, associated with the Laplace-Bessel differential operators. Trans. of NAS of Azerbj. 2006:71-78.

[10] Garcia-Cuerva J, Rubio de Francia J. Weighted norm inequalities and related topics. North Holland, Amsterdam; 1985.

[11] Ekincioglu I. The boundedness of high order Riesz-Bessel transformations generated by the generalized shift operator in weighted $L_{p,w,\gamma}$-spaces with general weights. Acta Appl.Math. 2010;109:591-598.

[12] Lyakhov LN. Multipliers of the mixed Fourier-Bessel transformation. Proc. V.A.Steklov Inst. Math. 2005;214:234-249.

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