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Structure of Some Pregroups and Length Functions

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The concept of Pregroups was introduced by Stallings in 1971. Subsequently the concept of Pregroups was developed by many other researchers. Stallings originally defined a set with a binary operation satisfying five axioms, namely, P1, P2, P3, P4, and P5. It has been proved later that P3 is a consequence of the other axioms. Stallings has also linked this construction of a Pregroup to Free Product of Groups.
This construction is developed to include a new axiom called P6, which enabled to define a length function on the universal group of Pregroups. Applications of Pregroups with length functions led to direct proof of many other problems in combinatorial group theory.

Keywords: Archimedean elements; defined product of elements; length functions; pregroup; universal group.

1 Introduction

Stallings [1], in 1971 introduced the concept of a pregroup. Subsequent work is done by Hoare [2], Nesayef [3], Chiswell [4], and many others. Five axioms are originally introduced by Stallings [1], namely P1, P2, P3, P4, and P5. It is proved in [3] that P3 is a consequence of the other axioms. Stallings extended his construction to link Pregroups to free products of groups in [5].

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On the other hand Lyndon [6] defined a length function on the elements of a group in 1963. Lyndon defined a set of axioms to be satisfied in order to assign a real valued length to each element of the given group. As a result of this development many other well-known theorems and properties of groups were proved directly by applying the length function. These were developed by many other researchers, such as Chiswell [3], Hoare [2], Wilkens [7] and many others. Nesayef [3] focussed on the applications of length functions on the universal group of pregroups. Further applications were carried out by Hoare [8] in 1988. Pregroups were also generalised further by Hoare [9] in 1992.

This paper considers imposing a new axiom called P6 and the pregroup satisfying P6 called P* and defining a length function on the universal group of P*. This result was also tackled independently by Chiswell [4]. Then combinatorial properties of groups with length functions are modified in this paper to include pregroups P*. As a result many new constructions are established and the nature of their elements are identified by the means of length functions.

In section one of this paper, we introduce the concept of length function and list all the axioms of length function which are needed in the latter sections. We also introduce the definition and some important properties of pregroups. In section two, we introduce the new axiom P6 and prove that some axioms are equivalent to the other ones.

Finally we show that the universal group of a Pregroup, U(P*) has a length function given by Lyndon [6] and prove some consequence results as a result of imposing P6.

2 Length Function

Definition 2.1: A length function | | on a group G, is a function giving each element x of G, a real number $|x|$, such that for all $x , y , z \in G$, the following axioms are satisfied.

$A1'$ $|e| = 0$, where e is the identity elements of G.
$A2$ $|x^{-1}| = |x|$
$A4$ $d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z)$, where $d(x,y) = \frac{1}{2}$ ($|x| + |y| - |xy^{-1}|$

Lyndon showed that A4 is equivalent to: $d(x,y) \geq \min\ \{ d(y,z), d(x,z)\}$ and to

$d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m$.
$A1', A2$ and $A4$ imply $|x| \geq d(x,y) = d(y,x) \geq 0$

Assuming, A2 and A4 only, it is easy to show that:

i. $d(x,y) \geq |e|$, where e is the identity element of G.
ii. $|x| \ \geq |e|$
iii. $d(x,y) \leq |x| - \frac{1}{2}\ |e|$

$A3$ State that $d(x,y) \geq 0$, is deductible from $A1', A2$ and $A1'$ is a weaker version of the following axiom:

$A1$ $|x| = 0$, if and only if $x = 1$ in G.

The following results are introduced by Lyndon [6].

(1) $d(xy,y) + d(x,y^{-1}) = |y|$
(2) $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$ Implies $|x\ y\ z| \leq |x| - |y| + |z|$
(3) $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$ Implies $d(xy,z^{-1}) = d(y,z^{-1})$
(4) $d(x,y) + d(x^{-1},y^{-1}) \geq |x| = |y|$ Implies $|(xy^{-1})^2| \leq |xy^{-1}|)$

It follows from (2), that for any $x, y \in G,\ d\,(x, y) = |y| - d\,(x\,y^{-1}, y^{-1}) \leq\ |y|$ by A3.

Since $d(x, y) = d\,(y\,, x)$, we get $d\,(x\,, y) \leq \min\{|x|\,, |y|\}$, as stated that $d\,(x\,, y) +\ d\,(x^{-1}\,, y^{-1}) > |x| = |y|\ \Rightarrow x = y$

The following definitions are introduced in [7].

Definition 2.2: A non-trivial element g of a group G is **called non-Archimedean** if $|g^2| \leq\ |g|$.

Definition 2.3: Let G be a group with length function. An element $x \neq 1$ in g is called Archimedean if $|x|\ \leq\ |x^2|$.

The following Axioms and results were added by Lyndon and others

$A0\quad x \neq 1\quad \Longrightarrow |x|\ <\ |x^2|$
$C0\ \ d(x\,, y\,)$ is always an integer
$C1\quad x \neq 1\ \ , |x^2|\ \leq\ |x|\ implies\ \ |x|$ is odd
$C2\ \ \ For\ \ no\ \ x\ is\ |x^2| = |x| + 1$
$C3\ \ \ if\ |x|\ is\ odd\ then\ |x^2| \geq\ |x|$
$C1'\ if\ |x|\ is\ even\ and\ |x| \neq 0\ , then\ |x^2| > |x|$
$N0\ \ |x^2| \leq |x|\ implies\ x^2 = 1\ \ is\ x = x^{-1}$
$N1^*\ G\ is\ general\ by\ \{x \in G\ :\ \ |x| \leq 1\}$

The following two constructions are also added by Lyndon [6], where the set of all Non-Archimedean elements in G is denoted by N:

$$N = \{x \in G\ :\ \ |x^2| \leq |x|\ \} \qquad (1)$$

$$M = \{\ xy\ \ \in G : |xy| +\ |yx| < 2|x| = 2\ |y|\ \} \qquad (2)$$

Lyndon showed that $M \subseteq N$. However, the nature of the elements of M and N is investigated in [10].

3 Pregroups

Definition 3.1. A Pregroup is a set P containing an element called the identity element of P, denoted by 1, a subset D of PXP and a mapping D → P, where (x , y) → x y, together with a map i : P → P where i (x) = x^{-1}, satisfying the following axioms:

We say that x y is defined if (x , y) ∈ D, i.e. x y ∈ P.

P1. For all x ∈ P, 1x and x1 are defined and 1x = x1 = x.
P2. For all x ∈ P, x^{-1} x = x x^{-1} = 1.
P3. For all x ,y ∈P, if x y is defined , then y^{-1} x is defined and $(x\ y\ )^{-1}$ = y x.
P4. Suppose that x, y, z ∈ P. If x y and y z are defined, then x (y z) is defined, is which case
x (y z) = (x y) z.
P5. If w, x, y, z ∈P, and if w x, x y, y z, are all defined the either w (x y) or (x y) z is defined.

3.1 The axiom P6

In this section we restrict our attention to a special type of pregroups, which satisfy a certain condition, namely P6. To do this we introduce the following theorems, which are given in [3].

Theorem 3.1: The following two statements are equivalent in P.

P6(1): If (x_1, x_2) is reduced and $x_1 a$, $a^{-1}x_2$ are both defined, then $a \in A_0$
P6(2): If (x_1, x_2) is reduced and $(ax_1)\, x_2$ is defined for $a \in P$ then $ax_1 \in A_0$.

Proof: Suppose (x_1, x_2) is reduced and let $(ax_1)\, x_2$ is defined for some $a \in P$.

$x_1(ax_1)^{-1}$, $(ax_1)\, x_2$ are both defined so $ax_1 \in A_0$, and P6 (1) → P6(2).

Conversely, suppose (x_1, x_2) is reduced and $x_1 a$, $a^{-1}x_2$ are both defined for some $a \in P$.

Since (x_1, x_2) is reduced, then $(x_1 a, a^{-1}x_2)$ is reduced.

Since $x_1^{-1}(x_1 a)$ is defined and equals to a, and $\{x_1^{-1}(x_1 a)\}\, a^{-1}x_2$ is defined and equals to x_2, then by P6 (2) $x_1^{-1}(x_1 a) \in A_0$, i. e. $a \in A_0$.

Therefore P6 (1) ⇔ P6 (2)

We denote the equivalent statements P6(1) and P6(2) in theorem 3.1, by P6 and the pregroup which satisfies P6, by P* . The following construction is introduced in [11].

Definition 3.2: Let P* be a pregroup satisfying P6. The Universal group, U (P*) is the set of all equivalence classes of reduced words in P*.

We define now a length function on $U(P^*)$. Before we achieve this, we introduce the following result, which generalizes the condition P6 (2).

Theorem 3.2: Let $a_{n-1}, \dots, a_1$ be any sequence, and $x_1, \dots, x_n$ be reduced, both on P^*. If $a_{n-1}, \dots, a_1 x_1, \dots, x_n$ is defined, then $a_{n-1}, \dots, a_1 x_1, \dots, x_n \in A_0$, $n \geq 2$.

Proof: The only way in which $a_{n-1}, \dots, a_1 x_1, \dots, x_n$ is defied is by $[a_{n-1}, \dots, a_1 x_1, \dots, x_n]x_n$ being defined.

Then also by theorem 2.1 either $[(a_{n-1}, \dots, a_1 x_1, \dots, x_{n-2})x_{n-1}]x_n$ is defied, so by P6(2), $(a_{n-1}, \dots, a_1 x_1, \dots, x_{n-2})x_{n-1} \in A_0$, or $[\, a_{n-1}(a_{n-2} \dots a_1 x_1 \;\dots\; x_{n-1})]x_n$ is defined, where $x_0 = 1$.

Since $(a_{n-2}, \dots, a_1 x_1, \dots, x_{n-1})x_n$ is not defined by theorem 2.1, then by P6(2) $a_{n-1}(a_{n-2} \;\dots\; a_1 x_1 \;\dots\; x_{n-1}) \in A_0$

Theorem 3.3: Let $U(P^*)$ be the universal group of a pergroup P^* and left $g, h \in (P^*)$. Let

$g = x_1 \dots x_n$, $h = y_1 \dots y_m , m, n \geq 2$ be in reduced forms. Let $a_i = x_{n-i+1} \dots .(x_n y_m^{-1}) \dots y_{m-i+1}^{-1}$ be defined for $1 \leq i \leq s$ for some $s < m , s \leq n$.

If $a_s\, y_{m-s}^{-1}$ is defined then $a_i \in A_0$ for all $i \leq s$. Hence by symmetry if $s < n$ and $x_{n-s} a_s$ is defined, then $a_i \in A_0$.

Proof: $a_i = x_{n-i+1}\, a_{i-1}\, y_{m-i+1}^{-1}$, for $1 \leq i < s$, where $a_0 = 1$. Then by theorem 2.1 either

$$x_{n-i+1}^{-1}(a_{i-1}\; y_{m-i+1}^{-1}) \text{ is defined} \tag{1}$$

$$\text{or } (x_{n-i+1}^{-1}\, a_{i-1})\, y_{m-i+1}^{-1} \text{ is defined} \tag{2}$$

If (1) holds, then $[x_{n-i+1}^{-1}(a_{i-1}\ y_{m-i+1}^{-1})]\ y_{m-i}^{-1}$ is defined

Then $a_{i-1}\ y_{m-i+1}^{-1}$, y_{m-i}^{-1} is reduced. So by P6 (2) $[x_{n-i+1}(a_{i-1}\ y_{m-i+1}^{-1})] = a_i \in A_0$

If (2) holds, then $[(x_{n-i+1}a_{i-1})\ y_{m-i+1}^{-1}]\ y_{m-i}^{-1}$ is defined.

Since y_{m-i+1}^{-1} , y_{m-i}^{-1} is reduced, then by P6 (2), $(x_{n-i+1}a_{i-1})\ y_{m-i+1}^{-1} = a_i \in A_0$

Corollary 3.1: In theorem 3.3, $x_{n-s}\,a_s\,(a_s\,y_{m-s}^{-1})$ is defined if and only if $(x_{n-s}\,a_s)\,a_s\,y_{m-s}^{-1}$ is defined.

Proof: By theorem 2.3, $a_s \in A_0$ in either case. Then $x_{n-s}\,a_s$ and $a_s\,y_{m-s}^{-1}$ are defined. by P4. Then the result follows.

From Theorems 2.2 and 2.3 and Corollary 2.1, with the same notations, we have shown that:

Corollary 3.2: $gh^{-1} = x_1 \dots . x_{n-s}a_s\,y_{m-s}^{-1} \dots . y_{m-s}^{-1}$ is reduced, if and only if $a_s \notin A_0$.

The proof of this Corollary is similar to the main theorem in [12].

3.2 P* pregroups and length function

Theorem 3.4: $|\ | : U(P^*) \rightarrow \mathbb{R}$ given in definition 3.2 is a length function on $U(P^*)$

Proof: $A1', A2$ are clearly satisfied, so we prove A4 is also satisfied.

Let $g, h, k \in U(P^*)$. The result is trivial if any one of $|g|, |h|, |k|$ is zero. So let $g = x_1 \dots . x_n, n \geq 1$, $h = y_1 \dots . y_m, m \geq 1$ and $z = z_1 \dots . z, z \geq 1$, be reduced where x_1, y_1 and $z_1 \notin A_0$,
i.e. $|g| = n, |h| = m\ and\ |k| = \ell$.

Clearly $d(g, h) \geq 0$. Suppose $d(g, h), d(h, k) \geq s$

Case 1: s is an integer

There exists a_s, such that, $gh^{-1} = x_1 \dots x_{n-s}\,a_s\,y_{m-s}^{-1} \dots y_1^{-1}$, where

$$a_s = x_{n-s+1} \dots x_n\,y_{m-s}^{-1} \dots y_{m-s+1},\ \text{and},\ a_s \in A_0$$

Similarly $hk^{-1} = y_1 \dots y_{m-s}\,b_s\,z_{\ell-s}^{-1} \dots z_1^{-1}$ and $b_s \in A_0$ $b_s = y_{m-s+1} \dots y_m\,z_\ell^{-1} \dots z_{\ell-s+1}^{-1}$

$$gh^{-1} = x_1 \dots x_n\,z_\ell^{-1} \dots z_1^{-1} = x_1 \dots x_n\,y_m^{-1} \dots y_{m-s+1}^{-1}\,y_{m-s+1}^{-1} \dots y_m\,z_\ell^{-1} \dots . z_1^{-1}$$
$$= x_1 \dots x_{n-s}\ a_s\,b_s\,z_{\ell-s}^{-1} \dots z_1^{-1}$$

Since $a_s\,b_s \in A_0$, then $d(g, k) \geq s$

Case 2: s is not an integer

Let $d(g, k), d(h, k) \geq r - \frac{1}{2} = s,\ r \geq 1$

Then $gh^{-1} = x_1 \dots x_{n-r}\ a_r\ y_{m-s}^{-1} \dots y_1^{-1}$, where $a_r = y_{n-r+1}^{-1} \dots x_n\,y_m^{-1} \dots y_{m-r+1}^{-1}$
$hk^{-1} = y_1 \dots y_{m-r}\ b_r\ z_{\ell-s}^{-1} \dots z_1^{-1}$, where $b_r = y_{m-r+1} \dots y_m$

$z_\ell^{-1} \dots z_{\ell-r+1}^{-1}$ and a_r , b_r are not necessarily in A_0, moreover a_i is defined for all $i \leq r$.

$$gh^{-1} = x_1 \;.\;\;.\;\;.\; x_n \; z_\ell^{-1} \;\ldots\; z_1^{-1} = x_1 \;.\;\;.\;\;.\; x_n \; y_{m-r+1}^{-1} \; y_{m-r+1} .\;\;.\;\;. y_m \; z_\ell^{-1} \;\ldots\; z_1^{-1}$$
$$= x_1 \;.\;\;.\;\;.\; x_{n-r} \, a_r \, b_r \; z_{\ell-r}^{-1} \;\ldots\; z_1^{-1}$$

Let $(a_r \, , b_r)$ be reduced. By theorem 2.3, $a_{r-1} \in A_o$.

Therefore $x_{n-r+1} \; a_{r-1}$ is defined and, $x_{n-r+1} \; a_{r-1} = ((x_{n-r+1} \; a_{r-1}) y_{m-r+1}^{-1}) y_{m-r+1} = a_r \; y_{m-r+1}$.

Similarly $y_{m-r+1}^{-1} \; b_r$ is defined, Then by $6(1)$, $y_{m-r+1} \in A_o$, so a contradiction.

Hence $a_r \; b_r$ is defined, thus $|gk^{-1}| \leq n - r + \ell - r + 1$, i.e. $d(g,k) \geq r - \frac{1}{2} = s$

Therefore A4 is satisfied, and so | | is a length function.

3.3 Applications of P* pregroups

Let $P = G \cup \{G \;\; t_i \;\; G\} \cup \{G \;\; t_j \;\; G\}$ and suppose that the product xy of two elements x and y of P is defined if and only if at least one of $\{x \, , y \, , xy\}$ is in G. Thus $x \, y \in A_0$, provided we exclude the case when

$$G^* = < c \, , t \, | rel \; c \; , \; tct^{-1} = \alpha \,(c) \, , t^2 = \; c' \in C >, \text{ in which case } A_0 = \; G^*$$

The axioms P1, P2 and P3 are clearly satisfied. For P4, let $x, y, z \in P$, and suppose xy, yz are defined, and x (y z) is also defined. If (x y) z is not defined, then

$$xy \notin G \, , z \;\notin G \text{ and } (xy) \; z \;\notin G \tag{1}$$

Thus

xy is defined ⇒ either x or $y \in G$ (2)

If yz is defined then either y or $yz \in G$ (3)

If X(yz) is defined then either x or $yz \; or \;\; x(yz) \in G$ (4)

Since $(xy)z \notin G$ and the products x(yz) and (xy)z are equal in G^*, therefore $x(yz) \notin G$

Case 1: If $x \in G$, then $yz \notin G$, so $y \in G$ by (3), and hence $xy \in G$, so a contradiction.

Case 2: If $x \notin G$, then $yz \in G$, so $y \in G$, by (2), and $yz \in G$ by (4). Therefore, $z \in G$ also a contradiction.

Hence (x y) z is defined, i.e P4 is satisfied.

For P5, let $w, x, y \; x \in P$ and let w x , x y, y z be defined and suppose neither w(xy) nor (xy)z is defined. Then $w \;\notin G \;\; , xy \;\notin G$ and $w \; (xy) \;\; \notin G$ since xy is defined. Then either $x \; or \; y \in G$.

Case 1: If $x \in G$ then x (yz) is defined, so by P4 (xy)z is also defined and x(yz) = (xy)z

Case 2: If $y \in G$ then (wx)y is defined , so by P4 w(xy) is defined hence P5 is satisfied.

For P6 suppose (x, y) is reduced, i.e non of the terms, x , y ,xy is in G, and suppose that $xa \, , a^{-1} \; y$ are both defined. Suppose also that $a \notin G$, since $x \notin G$, then $xa \in G$, and similarly $a^{-1}y \in G$.

Hence $(xa)(a^{-1}y) \in G$, so $xy \in G$ that is xy is defined, so a contradiction.

Therefore $a \in G$, and P6 is satisfied.

Note: The following cases are given in [10].

If $J = \emptyset$ and $N \neq \emptyset$, then we get an H.N.N extension, in which case $N \cap P^* = G$ and $M \cap A_0 = M \cap G =$ all elements of the associated subgroups.

If $J = \emptyset$, then we get a qusai- H.N.N. extension, (excluding $|J| = 1$, $I = \emptyset$), we have .

$N \cap P^* = G \cup \{g\,(t_j\, C_j)\}g^{-1} : g \in G$ $, j \in J$ and $M \cap A_0 = \{G_j\,,\ A_i\,,\ B_i\}$.

Theorem 3.5: If $\cup\ (P^*)$ is the universal group of a pergroup P^* then N consists of conjugates of elements of $P^* \cap N$. i.e. $g \in N \Leftrightarrow g = x\,a\,x^{-1}$, $x \in \cup\,(P^*)$ and, $a \in P^* \cap N$. Moreover if $|g|$ is even than $a \in A_0$ and $|x| = \frac{|g|}{2}$ and $|x| = \frac{|g|-1}{2}$, if $|g|$ is odd.

Proof: Suppose $g \in N$, and let $g = x_1 \ldots x_n$ be reduced, then $|g| = n$ or $g = 0$ if $n = 1$ and $x_1 \in A_0$.

The result is trivial if $n = 0\ or\ 1$, so let $n > 1$.

Case 1: If n is even, then put n = 2s, $s \geq 1$, $g^2 = x_1 \ldots x_s\, x_{s+1} \ldots x_n\, x_1 \ldots x_s x_{s+1} \ldots x_n$

Since $|g^2| \leq |g|$, then $g^2 = x_1 \ldots x_s\, x_{s+1} \ldots x_n$ $where$ $a_s = x_{s+1} \ldots x_n\, x_1 \ldots x_s \in' A_0$

By theorem 2.3, and since $|x_s\, a_s\, x_{s+1}| \leq 2$, then

$$g = x_1 \ldots x_s\, x_{s+1} \ldots x_n\, x_1 \ldots x_s\,(x_1 \ldots x_s)^{-1} = (x_1 \ldots x_s)\ a_s\,(x_1 \ldots x_s)^{-1}$$

= a conjugate of an element of A_0. Moreover, $a_s^2 \in A_0$, $i.e.$ $|a_s^2| = 0 = |a_s|$, so $a_s \in N$.

Case 2: If n is odd, then let $= 2r + 1$, $r \geq 1$.

$$g^2 = x_1 . \quad . \quad . x_r\, x_{r+1} . \quad . \quad . x_n\, x_1 . \quad . \quad . x_r\, x_{r+1}\, x_{r+2} . \quad . \quad . x_n$$

Since $|g^2| \leq |g|$, then

$g^2 = x_1 . \quad . \quad . x_r\,(x_{r+1}\, a_r\, x_{r+1})x_{r+2} . \quad . \quad . x_n,$ where $a_r x_{r+2} . \quad . \quad . x_n\, x_1 . \quad . \quad . x_r$ and $|x_{r+1}\, a_r\, x_{r+1}| \leq 1$.

By Theorem 2.2 , $a_r \in A_o$.

$$g = x_1 . \quad . \quad . x_r\, x_{r+1}\, x_{r+2} . \quad . \quad . x_n\, x_1 . \quad . \quad . x_r\,(x_1 . \quad . \quad . x_r)^{-1}$$
$$g = (x_1 . \quad . \quad . x_r)(x_{r+1}\, a_r)(x_1 . \quad . \quad . x_n)^{-1}, x_{r+1}\, a_r = b \in P * \backslash\ A_o.$$

Since $|x_{r+1}\, a_r\, x_{r+1}| \leq 1$ and since $a_r \in A_o$, then $b^2 = x_{r+1}\, a_r\, x_{r+1}\, a_r$ and $|b^2| \leq 1$, i.e. $|b^2| \leq |b|$. Hence b $\in P * \cap N$.

Conversely, suppose, g $= (x_1 . \quad . \quad . x_s)a(x_1 . \quad . \quad . x_s)^{-1}$, where $a \in P * \cap N$

If $x_s\ a\ x_s{}^{-1}$ is defined, then put $x_s\ a\ x_s{}^{-1} = a_1$, so $a \in A_o$, by theorem 2.3.

If $|a_1| = o$, then $|a_1^2| = o$, so $a_1 \in P * \cap N$.

Let $|a_1| = 1$. $a_1^2 = x_s\, a^2\, x_s^{-1}$, where $a^2 \in A_o$.

Suppose $(x_s\, a^2\, x_s^{-1})$ is reduced, i.e. $|a_1^2| = 2$. Apply P6(1) on $(x_s\, a^2\, x_s^{-1})$.

Since $(x_s\, a^2)(a^{-1}x_s^{-1})$ and $(x_s\, a^2\, x_s^{-1})$ are both defined, then $x_s\ \ a \in\ A_o$, so a contradiction.

Thus $x_s\, a^2\, x_s^{-1}$ is defined, i.e. $|a_1^2| \leq |a_1|$, so $a_1 \in\ P * \cap N$.

If $x_{s-1}\, a_1\, x_{s-1}^{-1}$ is also defined, then we apply the same argument and so on until we have

$g\ = (x_1 . \quad . \quad . \ x_r\,)\, b\, (x_1 . \quad . \quad . \ x_r\,)^{-1}$, where $b \in P * \cap N$, and $x_r\, b\, x_r^{-1}$ is not defined.

If $b \in\ A_o$, then $|g| = 2r$, $b^2 \in A_o$ and $|g^2| = |(x_1\ \dots\ x_s)\ b^2\,(x_1\ \dots\ x_s)^{-1}|\ \leq 2r$, so $g \in N$.

If $b\ \notin\ \ A_o$, and $x_r\ b$ and $b\ x_r^{-1}$ are not defined then $|g| = 2r + 1$.

Since $|b^2| \leq 1$, then $|g^2| = |(x_1\ \dots\ x_s)\ b^2\,(x_1\ \dots\ x_s)^{-1}| \leq 2r + 1 = |g|$, so $g \in N$.

Finally, if $b\ \notin A_o$ and either $x_r b$ or bx_r^{-1} is defined, then $|g| = 2r$.

Since $b \in N$ then $|b^2|\ \leq |b\,|$, so b^2 is defined.

Consider $g^2 = (x_1\ \dots\ x_s)\ b^2\,(x_1\ \dots\ x_s)^{-1}$, and suppose $x_r\,, b^2\,, x_r^{-1}$ is reduced. Then:

Either (i) bx_r^{-1} is defined, then apply P6 (2) on $(b^2\,, x_r^{-1})$.

Since $(b^{-1}\,, b^2\ )x_r^{-1} = b\, x_r^{-1}$ is defined, then $b^{-1} \in\ A_0$, so $b\ \in\ A_0$ and so a contradiction.

Or (ii) $x_r b$ is defined, and this is similar to (i)

Therefore $x_r\ \ , b^2\,, x_r^{-1}$ is not reduced, i.e. $|x_r\ \ \ b^2\ \ x_r^{-1}|\ \leq 2$, and so , $|g^2\ |\ \leq 2r =\ |g|$, $i.e\,.\,g\,\epsilon\, N$.

Theorem 3.6: The equivalent elements of N in $U(P^*)$ are the same conjugates of P^* i.e.

$|g| = |h|$, then $g \sim h\ \Leftrightarrow g = xax^{-1}\ \ , h =\ xbx^{-1}$ and $\sim b$, where $a, b \in P^*\,, x\ \in U\,(P^*)$

Proof: Suppose $g \sim h$ in N, then $|gh^{-1}| \leq\ |g| = |h|$

Let $g =\ x_1 \dots\ x_n\,, h =\ y_1\,\dots\ y_n$ be reduced.

Case 1: The result is trivial if $|g| = |h| = 0, 1$

Case 2: If n is even, then put $n = 2s\,,\ s \geq 1$

Then, $g =\ (x_1 \dots x_s)\, a_s\,(x_1 \dots x_s)^{-1}\,, a_s \in A_0$ and $h =\ (y_1 \dots y_s)\, b_s\,(y_1 \dots y_s)^{-1}\,,\ b_s \in\ A_{0.}$

$$gh^{-1} =\ \ (x_1 \dots x_{s-1})\,(x_s\, a_s)\,(x_1 \dots x_s)^{-1}\,(y_1 \dots y_s)\,(b_s^{-1}\ \ y_s^{-1})\,(y_1 \dots y_{s-1})^{-1}$$

Since $|gh^{-1}|\ \leq n = 2s$, then by theorem 2.3, $(x_1 \dots x_s)^{-1}\,(y_s \dots y_s) = a_s \in A_0$, and
$y_1 \dots y_s = (x_1 \dots x_s)\; a$

Thus $h = (x_1 \dots\ x_s)\ ab_s\, a^{-1}\,(x_1 \dots\ x_s)^{-1}$, where $ab_s\, a^{-1} = b \in A_0$

Hence g, h are the same conjugates, moreover, $|\ a_s\, b^{-1}| = 0$, for $a_s\,,\ b^{-1}\ \in A_0$

Case 3: If n is odd and $n = 2r + 1$ where $r \geq 1$, then we have:

$$g = (x_1 \dots x_r)(x_{r+1} a_r)(x_1 \dots x_r)^{-1}, a_r \in A_0, x_{r+1} a_r \in P^* \setminus A_0$$

Also $h = (y_1 \dots y_r)(y_{r+1} b_r)(y_1 \dots y_r)^{-1}, where\ b_r \in A_0, y_{r+1} b_r \in P^* \setminus A_0$

Since $|gh^{-1}| \leq 2r + 1$ then $|(x_{r+1} a_r)(x_1 \dots x_r)^{-1}(y_1 \dots y_r)(y_{r+1} b_r)^{-1}| \leq 1$.

Since $(x_{r+1} a_r)(x_1 \dots x_r)^{-1}$ and $(y_1 \dots y_r)(y_{r+1} b_r)^{-1}$ are reduced, it follows that:

$(x_1 \dots x_r)^{-1}(y_1 \dots y_r)$ is defined and by theorem 2.3 we have, $(x_1 \dots x_r)^{-1}(y_1 \dots y_r) = b \in A_0$

Thus $(y_1 \dots y_r) = (x_1 \dots x_r) b$.

So $h = (x_1 \dots x_r) b (y_{r+1} b_r) b^{-1}(x_1 \dots x_r)^{-1}$ where b $(y_{r+1} b_r) b^{-1} \in P^* \setminus A_0$.

Hence g and h are conjugates of elements of $P^* \setminus A_0$ determined by the same element of (P^*). Moreover $y_{r+1} a_r \sim b (y_{r+1} b_r) b^{-1}$, since $|x_{r+1} a_r b (y_{r+1} b_r) b^{-1}| \leq 1$ and $|b| = 0$

Conversely, suppose $g = (x_1 \dots x_r) a (x_1 \dots x_r)^{-1}, a \in P^* \cap N$, $h = (x_1 \dots x_r) b (x_1 \dots x_r)^{-1}, a \in P^* \cap N$, where $a \sim b$.

Similar arguments show that: $x_r a x_r^{-1}$ is defined where $r = \frac{|g|}{2}$ if $|g|$ is even, and $r = \frac{|g|-1}{2}$ if g is odd.

Since $a \sim b$, either $a, b \in A_0$ or $a, b \notin A_0$. So we consider the possible cases.

Case 1: If $a, b \in A_0$ then $ab^{-1} \in A_0$ and $|g| = |h| = 2r$.

$|gh^{-1}| = |(x_1 \dots x_r) b (x_1 \dots x_r)^{-1}| \leq 2r$, so $g \sim h$.

Case 2: If $a, b \notin A_0$ and both x_r, b, x_r^{-1} are reduced, then $|g| = |h| = 2r + 1$ and $|a| = |b| = 1$

Since $|ab^{-1}| \leq |a| = |b| = 1$ by assumption, then $|gh^{-1}| = |(x_1 \dots x_r) ab^{-1}(x_1 \dots x_r)^{-1}| \leq 2r + 1$, so $g \sim h$

Case 3: If $a, b \notin A_0$ and either $x_r a$ is defined (1)

Or $a x_r^{-1}$ is defined (2)

i.e. $|g| = 2r$ and either $x_r b$ is defined (3)

Or $b x_r^{-1}$ is defined (4)

i.e. $|g| = 2r$

Suppose $a x_r^{-1}$ is defined, and (x_r, a) is reduced, since $\in N$, then $|a^2| \leq 1$ and then $x_r a^{-1}$ and a^2 are both defined. So applying P6(1) to $(x_r a^{-1}, aa)$, then $a \in A_0$ so contradiction.

Hence $x_r a$ and $a x_r^{-1}$ are both defined. Similarly $x_r b$ and $b x_r^{-1}$ are both defined

So $|gh^{-1}| = |(x_1 \dots x_r a)(x_1 \dots x_r b)^{-1}| \leq 2r$. Therefore $g \sim h$.

Theorem 3.7. If $U(P^*)$ is the universal group of a pregroup $P^* \neq A_0$, then the elements of M are conjugates of elements of length zero in $U(P^*)$, i.e. $h \in M \implies gh = xax^{-1}$, where $x \in U(P^*)$ and $a \in A_0$

Proof: Let $g = x_1 \dots x_n$, $h = y_1 \dots y_n$ be reduced, and suppose $|gh| + |hg| < 2\,|h|$, $then\ |g|, |h| \geq 1$.

Case 1: If n = 1, then $g = x_1 \notin A_0$ and $h = h_1 \notin A_{0,}$ since $|x_1 y_1| + |y_1 x_1| < 2\,|x_1| = 2\,|y_2| = 2,$

Then at least one of $|x_1 y_1|$ or $|y_1 x_1|$ is zero

Suppose $|y_1 x_1| = 0$, i.e. $y_1 x_1 \in A_0$, then $gh = x_1 y_1 x_1 x_1^{-1} = x_1 (y_1 x_1) x_1^{-1}$ = conjugate of $y_1 x_1 \in A_{0.}$

Similarly, if $|x_1 y_1| = 0$, then hg is a conjugate of $x_1 y_1$.

Case 2: If $n \geq 2$, then let s be the maximum such that

$$gh = x_1 \dots x_{n-s}\, a_s\, y_{s+1} \dots y_n \tag{1}$$

$s \leq n$. Then either (1) is reduced in which case

$$|gh| = 2n - 2s + 1 \tag{2}$$

Or $a_s \in A_0$ and $x_{n-s}\, a_s\, y_{s+1}$ is not defined in which case

$$|gh| = 2n - 2s, \text{ where } a_s = x_{n-s+1} \dots x_n y_1 \dots y_s \tag{3}$$

Similarly, let r be the maximum such that

$$gh = y_1 \dots y_{n-r}\, b_r\, x_{r+1} \dots x_n \tag{4}$$

$$\text{Then either (4) is reduced, so } |hg| = 2n - 2r + 1 \tag{5}$$

$$\text{Or } b_r \in A_0 \text{ and } y_{n-r}\, b_r\, x_{r+1} \text{ is not defined, so } |hg| = 2n - 2r \tag{6}$$

Where $b_r = y_{m-r+1}\ y_m\, x_1 \dots x_r$

If (2) and (5) hold, then $2n - 2s + 1 + 2n - 2r + 1 < 2n$

$$2n - 2s - 2r + 2 < 0 \ \Rightarrow r + s > n - 1$$
$$r > n - s + 1 \text{ and } s > n - r + 1$$

If other cases hold, then it is clear that $r > n - s$ and $s > n - r$

Subcase 1: If (2) and (5) hold, then:

$$hg = (y_s \dots y_n)^{-1}\, y_s \dots y_n\, x_1 \dots x_{n-s+1}\ x_{n-s+2} \dots x_n\, y_1 \dots y_{s-1}$$

$$y_s \dots y_n = (y_s \dots y_n)^{-1}\, b_{n-s+1} \dots a_{s-1}\, (y_s \dots y_n)$$

Since $n - s + 1 < r,$ then $b_{n-s+1} \in A_0$, and since $a_{s-1} \in A_0$, then $b_{n-s+1}\, a_{s-1} \in A_0$. So g h is a conjugate of an element of A_0.

Subcase 2: If (3) and (5) or (6) hold, then

$$gh = (y_{s+1} \dots y_n)^{-1}\, y_{s+1} \dots y_n\, x_1 \dots x_{n-s}\ x_{n-s+1} \dots x_n$$
$$y_1 \dots y_s\, y_{s-1} \dots y_n = (y_{s+1} \dots y_n)^{-1}\, b_{n-s} \dots a_s\ (y_{s+1} \dots y_n)$$

Since $n - s < r$ then $b_{n-s} \in A_0$, and since $a_s \in A_0$, then $b_{n-s}\ a_s \in A_0$, then gh = conjugate of an element of $b_{n-s}\ a_s \in A_{0.}$

Subcase 3: If (2) and (6) hold, then

$$gh = (y_{n-r+1} \dots y_n)^{-1} y_{n-r+1} \dots y_n x_1 \dots x_r \; x_{r+1} \dots x_n \; y_1 \dots y_{n-r} y_{n-r+1} \dots y_n$$
$$= (y_{n-r+1} \dots y_n)^{-1} b_r \; a_{n-r} \; (y_{n-r+1} \dots y_n).$$

Since $b_r \; a_{n-r} \in A_0$ for $s > n-r,$ then g h = conjugate of an element of $b_r \; a_{n-r} \in A_0$.

Therefore, the elements of M are conjugate of zero length elements of $U(P^*)$

Since $n - s + 1 < r$ then $b_{n-s+1} \in A_0$, and since $a_{s-1} \in A_0$, then $b_{n-s+1} a_{s-1} \in A_0$, so gh is a conjugate of an element of A_0.

Subcase 4: If (3) and (5) or (6) hold, then

$$gh = (y_{s+1} \dots y_n)^{-1} y_{s+1} \dots y_n x_1 \dots x_{n-s} \; x_{n-s+1} \dots x_n \; y_1 \dots y_s y_{s+1} \dots y_n$$
$$= (y_{s+1} \dots y_n)^{-1} b_{n-s} \dots a_s \; (y_{s+1} \dots y_n)$$

Since $n - s < r$ then $b_{n-s} \in A_0$, and since $a_s \in A_0$ then $b_{n-s} \; a_s \in A_{0.}$

Therefore, g h = conjugate of an element of $b_{n-s} \; a_s \in A_0$

Subcase 5: If (2) and (6) hold, then

$$gh = (y_{n-r+1} \dots y_n)^{-1} y_{n-r+1} \dots y_n x_1 \dots x_r \; x_{r+1} \dots x_n$$
$$y_1 \dots y_{n-r} y_{n-r+1} \dots y_n = (y_{n-r+1} \dots y_n)^{-1} b_r \; a_{n-r} \; (y_{n-r+1} \dots y_n)$$

Since$b_r , a_{n-r} \in A_0$ for $s > n - r$ then, g h = conjugate of an element of $b_r \; a_{n-r} \in A_0$.

Therefore, the elements of M are conjugate of zero length elements of $U(P^*)$.

4 Conclusion

This paper shows that a special type of pregroups which satisfy an additional condition namely P6 can be occupied with Length Function defined by Lyndon [6]. Therefore, it will have all the combinatorial group properties which are open for investigation. This paper also proved the following:

(1) The elements of N consists of conjugates of elements of $P^* \cap N$.
(2) The equivalent elements of N in $U(P^*)$ are the same conjugates of P^*.
(3) The elements of M are conjugate of elements of length zero in $U(P^*)$.

Competing Interests

Author has declared that no competing interests exist.

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