Adomain Sumudu Transform Method for the Blasius Equation

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Authors' contribution

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Original Research Article

Abstract

This paper presents a new method namely, Adomain Sumudu Transform Method, a coupling of the Sumudu transform and Adomain decomposition method, for handling a differential equation of mixing layer that arises in viscous incompressible fluid. In order to apply the condition at infinity, we converted the obtained series solution into rational function by using Padé approximant.

Keywords: Sumudu transform; Adomain decomposition; Padé approximant; Blasius equation.

1 Introduction

Blasius equation is one of the basic equations in fluid dynamics and it describes steady flow of viscous incompressible fluids over a semi-infinite flat plate [1]. Its first appearance in the literature was recorded in 1908 [2]. As a result of the application of Blasius equation to fluid flow, engineers, physicists and mathematicians have special interest in studying the equation and the related equations with boundary conditions at infinity, especially Falkner-Skan equation. There are two

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forms of the Blasius equation; both forms are represented by the same differential equation but with different boundary conditions [3, 4, 5]. The existence of a solution for the Blasius equation was considered and established by [6] using Weyl technique [7]. Because of the challenge posed by the boundary condition at infinity, authors have suggested various ways of overcoming this difficulty. Such ways include changing the boundary conditions at infinity into a classical conditions [8], converting the Blasius equation(a boundary value problem) to a pair of initial value problems [9, 10] and then solving the pair of initial value problems. Another possible way to do this is by applying the Padé approximation technique if the solution obtained is a series. Many numerical, analytical and semi-analytical methods have been investigated for solving this equation. Some of these methods are finite difference, Adomain decomposition, perturbation methods, differential transform and variation iteration methods. Inspite of the numerous efforts however, a truly simple, yet numerically accurate algorithm is still missing. Therefore, this paper seeks to develop a simple and reliable method for solving the Blasius equation of the form considered by [3, 4, 11]. We shall use the Padé approximants to handle the boundary conditions at infinity.

2 Some Preliminaries

In this section, we present a brief review three methods which serve as the building blocks for the proposed method.

2.1 Sumudu transform

Among the common integral transforms in the literature that are widely used in physics, astronomy and engineering are Fourier, Laplace, Hankel and Mellin Transform. However, Sumudu transform was introduced in 1993 by Watugala [12] to solve differential equations and control engineering problems. Since then, several authors have studied its properties, application to solving various problems and its relationship with some other common integral transforms such as Laplace transform [12, 13, 14, 15]. Apart from some other advantages of Sumudu transform over other integral transforms such as simplicity and accuracy, Zhang [16] noted that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except a factor of $n!$

The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(u)$, defined by

$$S\left(f(t)\right) = F(u) = \int_0^\infty \frac{1}{u} e^{\left(\frac{-t}{u}\right)} f(t)dt, \tag{2.1}$$

where the symbol S denotes the Sumudu transform.

Few basic properties of the Sumudu transform:

If c_1, c_2 are non-negative constants, $f(t)$ and $g(t)$ are functions having Sumudu transform $F(u)$ and G(u), respectively, then

1. Linearity Property

$$S\left[c_1 f(t) + c_2 g(t)\right] = c_1 S\left[f(t)\right] + c_2 S\left[g(t)\right] \tag{2.2}$$

2. Convolution Property

$$S\left[(f * g)(t)\right] = uS\left[f(t)\right] + S\left[g(t)\right] \tag{2.3}$$

3. Differentiation Property

$$S\left[f^{(n)}(t)\right\} = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right] \tag{2.4}$$

2.2   Adomain decomposition method

In recent years, Adomain decomposition method(ADM) has been applied to solve a wide range of linear and nonlinear differential, integral and integro-differential equations [17, 18, 19]. Unlike many other methods, ADM provides solution to nonlinear problems without linearization, perturbation or discretization in form of a convergent series. We now illustrate the basic principles of ADM.

Consider a general functional equation

$$Lu + Ru + Nu = g(t), \tag{2.5}$$

where $u(t)$ is the unknown function and the linear terms are decomposed into $L+R$ and Nu denotes the nonlinear terms. L is usually the highest order differential operator and invertible, R is the remainder of the linear operator and $g(t)$ is the source term.

Applying the inverse linear operator L^{-1} to both sides of (2.5) gives

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2.6}$$

If L is a second-order linear differential operator, L^{-1} is a two-fold integral operator, then we get

$$u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{2.7}$$

where A and B are the constants of integration and can be found from the initial or boundary conditions. ADM decomposes the solution into a series

$$u = \sum_{n=0}^{\infty} U_n, \tag{2.8}$$

and decomposes the nonlinear terms Nu into a series

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{2.9}$$

where A_n are the Adomain polynomials. Substituting (2.8) and (2.9) into (2.7), we obtain the solution

$$u_n = A + Bt + L^{-1}g - L^{-1}\left[R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right], \tag{2.10}$$

where the Adomain polynomial A_n can be generated by using the formula [17].

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \qquad n = 0, 1, \cdots. \tag{2.11}$$

The solution components $u_n(x)$ may be determined by using the classic Adomain recursive scheme as follows:

$$u_0 = A + Bt + L^{-1}g, \tag{2.12}$$

$$u_{n+1} = -L^{-1}\left[Ru_n + A_n\right], n \geq 0. \tag{2.13}$$

2.3 Pad*é* approximant

Often time, power series don't give a good approximation to a function except the radius of convergence is sufficiently large to contain the domain $[a, b]$ over which the function is approximated. In order to make the maximum error as small as possible, a rational(Pad*é*) approximation method which has a smaller error on $[a, b]$ than a polynomial approximation can be constructed. Pad*é* approximation to $f(x)$ on $[a, b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M, respectively. We use $[N/M]$ to denote this quotient [20] such that

$$[N/M] = \frac{P_N(x)}{Q_M(x)}, \quad a \le x \le b. \tag{2.14}$$

The polynomials $P_N(x)$ and $Q_M(x)$ are constructed in such a way that $f(x)$ and the Pad*é* approximant as well as their derivatives up to $N + M$ agree at $x = 0$. Assume that $f(x)$ is analytic and has Maclaurin series expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots, \tag{2.15}$$

and form the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) \left(1 + \sum_{i=1}^{M} q_i x^i \right) - \left(\sum_{i=0}^{N} p_i x^i \right) = \left(\sum_{i=N+M+1}^{\infty} c_i x^i \right) \tag{2.16}$$

Expanding (2.16) and equating the coefficients of powers of x^i to zero for $i = 0, 1, \cdots, N + M$, produces $N + M + 1$ linear equations which are solved to determine the values of the coefficients $q_1, q_2, \cdots, q_M, p_0, p_1, \cdots, p_N$.

For a fixed value of $N + M$ the error in the approximation is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

3 Analysis of Adomain Sumudu Transform Method (ASTM)

Consider the Blasius equation

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, 0 < \eta < \infty \tag{3.1}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0. \tag{3.2}$$

Taking the Sumudu transform of both sides of (3.1), we have

$$S\left[f'''(\eta)\right] = -\frac{1}{2} S\left[f(\eta) f''(\eta)\right] \tag{3.3}$$

Applying the differentiation property of Sumudu transform to the LHS of (3.3), we have

$$\frac{1}{u^3} \left[S\left[f(\eta)\right] - f(0) - u f'(0) - u^2 f''(0) \right] = -\frac{1}{2} S\left[f(\eta) f''(\eta)\right] \tag{3.4}$$

Substituting the initial conditions, with the assumption that $f''(0) = \alpha$, we obtain

$$S\left[f(\eta)\right] = u + \alpha u^2 - \frac{1}{2} u^3 S\left[f(\eta) f''(\eta)\right] \tag{3.5}$$

Taking the inverse Sumudu on both sides of equation (3.5), we find

$$f(\eta) = \eta + \frac{1}{2!}\alpha\eta^2 - \frac{1}{2}S^{-1}\left[u^3 S\left[f(\eta)f''(\eta)\right]\right] \tag{3.6}$$

The Sumudu decomposition method assumes a series solution of the function $f(\eta)$ given by

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta), \tag{3.7}$$

so that (3.6) becomes

$$\sum_{n=0}^{\infty} f_n(\eta) = \eta + \frac{1}{2!}\alpha\eta^2 - \frac{1}{2}S^{-1}\left[u^3 S\left(\sum_{n=0}^{\infty} A_n\right)\right], \tag{3.8}$$

where

$$\sum_{n=0}^{\infty} A_n = f(\eta)f''(\eta), \tag{3.9}$$

and the Adomain polynomials A_n are calculated by using the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i f_i\right)\left(\sum_{i=0}^{\infty}\lambda^i f_i''\right)\right]_{\lambda=0}, \quad n=0,1,\cdots. \tag{3.10}$$

The first few Adomain polynomials are obtained as follows:

$$A_0 = f_0(\eta)f_0''(\eta) \tag{3.11}$$

$$A_1 = f_0(\eta)f_1''(\eta) + f_1(\eta)f_0''(\eta) \tag{3.12}$$

$$A_2 = f_0(\eta)f_2''(\eta) + f_1(\eta)f_1''(\eta) + f_2(\eta)f_0''(\eta) \tag{3.13}$$

$$A_3 = f_0(\eta)f_3''(\eta) + f_1(\eta)f_2''(\eta) + f_2(\eta)f_1''(\eta) + f_3(\eta)f_0''(\eta) \tag{3.14}$$

$$A_4 = f_0(\eta)f_4''(\eta) + f_1(\eta)f_3''(\eta) + f_2(\eta)f_2''(\eta) + f_3(\eta)f_1''(\eta) + f_4(\eta)f_0''(\eta) \tag{3.15}$$

$$A_5 = f_0(\eta)f_5''(\eta) + f_1(\eta)f_4''(\eta) + f_2(\eta)f_3''(\eta) + f_3(\eta)f_2''(\eta) + f_4(\eta)f_1''(\eta) + f_5(\eta)f_0''(\eta) \tag{3.16}$$

We now take

$$f_0(\eta) = \eta + \frac{1}{2!}\alpha\eta^2 \tag{3.17}$$

as the first approximation to $f(\eta)$ in equation (3.6), and the higher iterates of $f(\eta)$ are obtained from the recurrence relation

$$f_{n+1}(\eta) = -\frac{1}{2}S^{-1}\left[u^3 S\left(\sum_{n=0}^{\infty} A_n\right)\right], n \geq 0. \tag{3.18}$$

Thus, we have

$$f_1(\eta) = -\frac{1}{48}\alpha\eta^4 - \frac{1}{240}\alpha^2\eta^5, \tag{3.19}$$

$$f_2(\eta) = \frac{1}{960}\alpha\eta^6 + \frac{11}{20160}\alpha^2\eta^7 + \frac{11}{161280}\alpha^3\eta^8, \tag{3.20}$$

$$f_3(\eta) = -\frac{1}{21504}\alpha\eta^8 - \frac{43}{967680}\alpha^2\eta^9 - \frac{5}{387072}\alpha^3\eta^{10} - \frac{1}{4257792}\alpha^4\eta^{11}. \tag{3.21}$$

Therefore, the solution of (3.1) is in a series form given by

$$f(\eta) = \eta + \frac{1}{2}\alpha\eta^2 - \frac{1}{48}\alpha\eta^4 - \frac{1}{240}\alpha^2\eta^5 + \frac{1}{960}\alpha\eta^6 + \frac{11}{20160}\alpha^2\eta^7 + \left(\frac{11\alpha^3}{161280} - \frac{\alpha}{21504}\right)\eta^8$$

$$- \frac{43}{967680}\alpha^2\eta^9 - \frac{5}{387072}\alpha^3\eta^{10} - \frac{1}{4257792}\alpha^4\eta^{11} + \cdots, \tag{3.22}$$

and so

$$f'(\eta) = 1 + \alpha\eta - \frac{1}{12}\alpha\eta^3 - \frac{1}{48}\alpha^2\eta^4 + \frac{1}{160}\alpha\eta^5 + \frac{11}{2880}\alpha^2\eta^6 + \left(\frac{11}{20160}\alpha^3 - \frac{1}{2688}\right)\eta^7$$

$$- \frac{43}{107520}\alpha^2\eta^8 - \frac{25}{193536}\alpha^3\eta^9 - \frac{5}{387072}\alpha^4\eta^{10} + \cdots \tag{3.23}$$

We are now to determine the value of the constant α by using the condition at ∞, that is, $f(\infty) = 0$. However, since it is impossible to apply this condition directly to (3.23), we obtain rational function representations of (3.23) by using the Padé approximants. Following the illustration given in section 2.3, the Padé approximant [2/2] to $f'(\eta)$ of degree 4 from equation (3.23) is obtained as

$$[2/2] = \frac{1 + \frac{3}{4}\alpha\eta + \left(\frac{1}{12} - \frac{1}{4}\alpha^2\right)\eta^2}{1 - \frac{1}{4}\alpha\eta + \frac{1}{12}\eta^2}. \tag{3.24}$$

Applying the condition $f(\infty) = 0$ yields

$$\alpha = 0.5773502692. \tag{3.25}$$

In a similar manner, for Padé approximant [3/3] to $f'(\eta)$ of degree 6 we obtain

$$\alpha = 0.5163977795, \tag{3.26}$$

and likewise Padé approximant [4/4] gives

$$\alpha = 0.5227030798. \tag{3.27}$$

Comparison of Numerical values of $\alpha = f''(0)$

Padé Approximant	ASTM	VIM [3]	MADM[4]
[2/2]	0.5773502692	0.5773502692	0.5773502693
[3/3]	0.5163977795	0.5163977793	0.5163977793
[4/4]	0.5227030798	0.5227030796	0.5227030798

4 Conclusion

We employed the combination of Sumudu transform and Adomian decomposition method to obtain a closed form solution of the Blasius equation. The new method is free of unnecessary mathematical complexities. Although the problem considered has no exact solution, the accuracy and reliability of the new method are guaranteed because the results obtained are in complete agreement with those obtained by powerful methods like modified Adomain decomposition(MADM) and variation iteration method(VIM).

Competing Interests

Authors have declared that no competing interests exist.

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