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A Distance Measure Based on Fuzzy D-implications: Application in Pattern Recognition

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Authors' contributions

This work was carried out in collaboration between all authors. Author AGH designed the study, wrote the protocol and supervised the work. Author GAP managed the analyses of the study and performed the experiments. Authors AGH and VGK managed the mathematical content and the literature searches. All authors read and approved the final manuscript.

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Abstract

A new distance measure between fuzzy sets (FSs) based on fuzzy D-implications is introduced in this paper. The proposed measure uses a matrix representation of each set in order to encode its information, where matrix norms in conjunction with fuzzy D-implications can be applied to measure the distance between the two FSs. It is worth noting that the applied technique in deriving the proposed measure gives the flexibility to construct several distance measures by incorporating different fuzzy implications, extending its applicability to several applications where the most appropriate implication is used. Apart from the analysis in constructing a D-implication based distance measure, a detailed discussion of its main properties is also presented. Moreover, an appropriate set of experiments has taken place in order to examine the performance of the proposed distance compared to well-known fuzzy implications, in some pattern classification problems from the literature. The corresponding results are promising and show that the proposed measure can classify the patterns correctly and with high degree of confidence.

Keywords: Fuzzy sets; fuzzy implications; distance measure; pattern recognition.

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1 Introduction

An increased interest in proposing efficient distance measures between Fuzzy Sets (FSs) of several types [1-3] has been reported in the literature in the last decades. The scientific interest is mainly focused on developing distance measures able to measure accurately the distance between two fuzzy sets/numbers according to specific principles and definitions.

Among these measures, there is a newly introduced type of distance measures [4,5] that make use of fuzzy implications in a matrix formulation in order to compute the distance between two fuzzy sets, with satisfactory performance. However, although the *S*-, *R*-, and *QL* - implications have been used extensively, D-implications are less used and reported in the literature. This fact constitutes a particular motivation to investigate and analyze some of D-implications properties by constructing a concrete theoretical framework, able to allow the definition of a distance measure.

In this context, apart from the theoretical analysis, the construction of a distance measure based on D-implication is proposed hereafter and is used to classify known patterns in three pattern classification problems. Finally, its performance is compared with that of well-known fuzzy implications of the literature, under several experimental configurations.

We remark that a preliminary work [6] has focused on presenting main properties of D-implications. This paper extends the work in [6] by, first, defining a novel distance measure based on D-implications and, second, by studying comparatively the effectiveness of the proposed distance measure in a set of experiments regarding pattern classification problems.

The paper is organized by presenting some mathematical preliminaries on Fuzzy Implications, with emphasis on D-implications in Section 2. Moreover, Section 3 describes in detail the applied technique to derive a distance measure based on D-implications and analyses its main properties, while its application to pattern classification problems along with a comparison with other popular implications is presented in Section 4. Finally, Section 5 summarizes the main conclusions derived by the overall experimental study.

2 Fuzzy Implications - Definitions and Notations

2.1 Fuzzy implications. Basic notations

A *fuzzy implication* I is a function of the form: $I:[0,1]\times[0,1]\rightarrow[0,1]$ where for any possible truth values a, b of given fuzzy propositions p, q, respectively, it defines the truth value, $I(a,b)$, of the conditional proposition "if p then q ". The function $I(.,.)$ should be an extension of the classical implication from domain $\{0,1\}$ to the domain $[0,1]$, of truth values in fuzzy logic.

The *implication operator* of the classical logic is a mapping: $m:\{0,1\}\times\{0,1\}\rightarrow\{0,1\}$, which satisfies the conditions: $m(0,0)=m(0,1)=m(1,1)=1$ and $m(1,0)=0$. These conditions are the least ones that we can demand from a fuzzy implication operator. In other words, fuzzy implications collapse to the classical implication, when the truth values are restricted to 0 and 1; i.e. $I(0,0)=I(0,1)=I(1,1)=1$ and $I(1,0)=0$.

One way to define m in classical logic is to use the logic formula $\forall a,b\in\{0,1\}$:

$$m(a,b)=\bar{a}\vee b \tag{1}$$

Another two different Boolean expressions for the implication function m are shown next.

$$m(a,b)=\max\ \left\{x\in\{0,1\}\,|\,\left(a\wedge x\right)\leq b\right\} \tag{2}$$

$$m(a,b)=\bar{a}\vee(a\wedge b) \tag{3}$$

The extensions of these equations in fuzzy logic are $\forall a,b\in[0,1]$, respectively,

$$I_S\left(a,b\right)=S\left(n\left(a\right),b\right) \tag{4}$$

$$I_R\left(a,b\right)=\sup\left\{x\in[0,1]\,|\,T\left(a,x\right)\leq b\right\} \tag{5}$$

$$I_{QL}\left(a,b\right)=S\left(n\left(a\right),T\left(a,b\right)\right) \tag{6}$$

Where T is a t-norm, S is a t-conorm, n is a fuzzy negation, on $[0,1]$ and the triple $<T,S,n>$ is required to satisfy the De Morgan laws.

The fuzzy implications obtained from Eq. (4) are usually referred to in the literature as *S-implications*. Moreover, the fuzzy implications obtained from Eq. (5) are called *R-implications*, while those obtained from Eq. (6) are called *QL-implications* [7-10].

In the following, when reporting to *S*-, *R*-, and *QL*- implications we will mean fuzzy implications with types (4), (5), (6), respectively.

2.2 Fuzzy D-implications

In addition to the three classes of fuzzy implications (*S*-, *R*-, *QL*-implications), which are predominant in the literature, other fuzzy implications are possible [7].

The formula $m\left(a,b\right)=\overline{a}\vee b$, $\forall a,b\in\{0,1\}$ may also be rewritten, due to the law of absorption of negation in classical logic, as:

$$m\left(a,b\right)=\left(\overline{a}\wedge\overline{b}\right)\vee b,\forall a,b\in\{0,1\} \tag{7}$$

The extension of this equation in fuzzy logic is:

$$I\left(a,b\right)=S\left(T\left(n\left(a\right),n\left(b\right)\right),b\right),\forall a,b\in[0,1] \tag{8}$$

Where *S* and *T* are dual with respect to n . In the literature, these fuzzy implications are called *Dishkant* implications or D-implications in short, they will be symbolized $I_D\left(.,.\right)$ throughout this text and is called an I_D*-implication* [11].

The following Propositions give some properties and characteristics of I_D-implications [6]. Their proofs are shown in Appendix. These properties are also required for some *S*-, *R*- and *QL*-implications [12] and could be important in specific applications.

Proposition 1. Let fuzzy implication I_D, then the following properties hold, $\forall a,b,x \in [0,1]$.

i) $a \le b \Rightarrow I_D(x,a) \le I_D(x,b) \Leftrightarrow I_D(a, x \wedge b) = \wedge(I_D(a,x), I_D(a,b))$.

ii) $a \le b \Rightarrow I_D(x,a) \le I_D(x,b) \Leftrightarrow I_D(a, x \vee b) = \vee(I_D(a,x), I_D(a,b))$.

The proof of Proposition 1 is shown in the Appendix.

Let *T* be a *t*-norm, *S* a *t*-conorm and *n* a strong negation. Then, the corresponding *QL*-operator, I_{QL}, is a *QL*-implication if and only if the corresponding *D*-operator, I_D, is a *D*-implication [11]. So, in general I_D-implications violate property $a \le b \Rightarrow I_D(a,x) \ge I_D(b,x)$. The conditions under which this property is satisfied from I_{QL}-implications (therefore and from I_D-implications) can be found in [13].

Proposition 2. Let I_D be a fuzzy D-implication such that satisfy the property $a \le b \Rightarrow I_D(a,x) \ge I_D(b,x)$, $\forall a,b,x \in [0,1]$. Then the following properties hold, $\forall a,b,x \in [0,1]$.

i) $a \le b \Rightarrow I_D(a,x) \ge I_D(b,x) \Leftrightarrow I_D(a \wedge x, b) = \vee(I_D(a,b), I_D(x,b))$.

ii) $a \le b \Rightarrow I_D(a,x) \ge I_D(b,x) \Leftrightarrow I_D(a \vee x, b) = \wedge(I_D(a,b), I_D(x,b))$.

The proof of Proposition 2 is shown in the Appendix.

Proposition 3. Let fuzzy implication $I_D(a,b) = \vee(\wedge(n_S(a), n_S(b)), b)$, where $n_S(a) = 1 - a$, then the following properties hold, $\forall a,b,c \in [0,1]$.

i) $I_D((a \wedge b), c) \ge I_D((a \vee b), c)$.

ii) $\wedge(I_D(a,c), I_D(b,c)) \le I_D(\wedge(a,b), c) = \vee(I_D(a,c), I_D(b,c))$.

iii) $\wedge(I_D(a,c), I_D(b,c)) = I_D(\vee(a,b), c) \le \vee(I_D(a,c), I_D(b,c))$.

iv) $\vee(a, I_D(a,b)) = I_D(a, \vee(a,b))$

v) $\wedge(I_D(a,c), I_D(b,c)) \le I_D(a,b)$, for $a \le \mathrm{c} \le b$

The proof of Proposition 3 is shown in the Appendix.

The classical modus ponens is the tautology: $(a \wedge (a \Rightarrow b)) \Rightarrow b$. Modus ponens states that given two true propositions, "a" and "$a \Rightarrow b$", the truth of the proposition "b" may be inferred.

In fuzzy logic, after algebraic interpretation in terms of truth values, this becomes as follows: The truth value of "a" and "$a \Rightarrow b$" must be less than or equal to the truth value of "b", which can be expressed as $T\left(a, I\left(a,b\right)\right) \le b$, $\forall a,b \in \left[0,1\right]$, where is T a t-norm and I is a fuzzy implication.

For fuzzy D-implication $I_D\left(a,b\right) = \vee\left(\wedge\left(n_S\left(a\right), n_S\left(b\right)\right), b\right), a,b \in \left[0,1\right]$, where $n_S\left(a\right) = 1 - a$, it holds:

$\wedge\left(a, I_D\left(a,b\right)\right) \le b$, for $a \le b$ (the proof is straightforward).

3 A Distance Measure

3.1 Metric distance - definitions and basic notations

Definition 1. A *metric distance* in a set X is a real function $d : X \times X \rightarrow R$ which satisfies ($\forall x, y, z \in X$):

a) $d(x,y) = 0 \Leftrightarrow x = y$,
b) $d(x,y) = d(y,x)$, *symmetric*,
c) $d(x,z) + d(z,y) \ge d(x,y)$, *Triangle Inequality*.

Various distance measures have previously been proposed in the literature involving fuzzy sets [1,2,14,15]. Some common metrics used to describe the distance between fuzzy sets are the following Eq. (9-12).

If the universe set E is finite, i.e., $X = \left\{x_1, \ldots, x_n\right\}$ then for any two fuzzy subsets A and B of X with membership functions $\mu_A\left(.\right)$ and $\mu_B\left(.\right)$, respectively, we have:

Hamming distance
$$d_H\left(A,B\right) = \sum_{i=1}^{n}\left|\mu_A\left(x_i\right) - \mu_B\left(x_i\right)\right| \tag{9}$$

Normalized Hamming distance
$$d_{n-H}\left(A,B\right) = \frac{1}{n}\sum_{i=1}^{n}\left|\mu_A\left(x_i\right) - \mu_B\left(x_i\right)\right| \tag{10}$$

Euclidean distance
$$d_E\left(A,B\right) = \sqrt{\sum_{i=1}^{n}\left(\mu_A\left(x_i\right) - \mu_B\left(x_i\right)\right)^2} \tag{11}$$

Normalized Euclidean distance
$$d_{n-E}\left(A,B\right) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A\left(x_i\right) - \mu_B\left(x_i\right)\right)^2} \tag{12}$$

3.2 Proposed distance measure

A new family of normalized distance measures between fuzzy sets based on matrix norms and fuzzy implications has been suggested in [4] and extended for the case of intuitionistic fuzzy sets in [5].

Furthermore, it is remarked [16] that if $\Pi_1 = (a_{ij}), \Pi_2 = (b_{ij})$, $i = 1,\dots,n$, $j = 1,\dots,n$ are square matrices then the norm $\|\cdot\|$ can be used to define a metric d as:

$$d(\Pi_1, \Pi_2) = \|\Pi_1 - \Pi_2\| \tag{13}$$

Let A be fuzzy set in a finite universe $X = \{x_1,\dots,x_n\}$, with membership function $\mu_A(.)$. Let I_D be a fuzzy D-implication. We define the $n \times n$ matrices $\Pi(\mu_A)$ of I_D as follows:

$$\Pi(\mu_A) \triangleq \left[\underset{i=1,\dots,n}{I_D}\left(\mu_A(x_i), \mu_A(x_i)\right)\right] = I_D\left(\begin{bmatrix}\mu_A(x_1)\\ \vdots \\ \mu_A(x_n)\end{bmatrix}, \left[\mu_A(x_1),\dots,\mu_A(x_n)\right]\right)$$

$$= \begin{bmatrix} I_D\left(\mu_A(x_1), \mu_A(x_1)\right) & \dots & \dots & I_D\left(\mu_A(x_1), \mu_A(x_n)\right) \\ \vdots & \vdots & \vdots & \vdots \\ I_D\left(\mu_A(x_n), \mu_A(x_1)\right) & \dots & \dots & I_D\left(\mu_A(x_n), \mu_A(x_n)\right) \end{bmatrix}$$

Let X denote a universe of discourse, where X is a finite and let $F(X)$ denote the set of all fuzzy sets in X.

Definition 2. Given two fuzzy sets $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$, $B = \{\langle x, \mu_B(x)\rangle \mid x \in X\}$, where $X = \{x_1,\dots,x_n\}$ is a finite universe of discourse. Also, let I_D be a fuzzy D-implication and any tensor-or operator-norm $\|\cdot\|$. Then

$$d(A, B; I_D) \triangleq \left\|\Pi(\mu_A) - \Pi(\mu_B)\right\| \tag{14}$$

where $\Pi(\mu_.) = \left[\underset{i=1,\dots,n}{I_D}\left(\mu_.(x_i), \mu_.(x_i)\right)\right]$, defines a metric distance

$$d : F(X) \times F(X) \to [0, +\infty).$$

The above function $d(A, B; I_D)$ is a metric [4]. So, this definition actually introduces multiple families of metrics with different meanings, according to the binary operator chosen.

In Eq. (14) the norm $\|\Pi\|$ is computed by using the largest non negative eigenvalue of the positive definite Hermitian matrix $\Pi^T \Pi$ (Π^T is the transpose of matrix Π) [16],

$$\|\Pi\| = \sqrt{\lambda_{\max}} \tag{15}$$

4 Application in Pattern Classification

In order to study the ability of the proposed distance measure to count the distance between two fuzzy sets, a set of experiments have been conducted. For this purpose, three well known from the literature problems, according to which a test sample need to be classified to a specific category, are selected.

In the following examples, attributes correspond to the measurements that are used to describe each class, while the classes are represented by specific patterns that they describe the classes' centroids. This procedure constitutes the main operation of the minimum-distance classifier, where the test sample is assigned to the class from which its distance is minimum and is described by the following equation:

$$k^{*}=\arg\min_{k}\left\{Dist\left(P_{k},S\right)\right\} \tag{16}$$

The proposed D-implication based distance measure is compared with similar distances derived by using the popular *engineering implications* of *Mamdani* ($\sigma_{\mathrm{M}}\left(a,b\right)=\min\left\{a,b\right\}$) and *Larsen* ($\sigma_{\mathrm{L}}\left(a,b\right)=ab$) commonly used in Fuzzy Inference Systems (FIS), noted as d_M and d_L respectively. Furthermore, in order to study the ability of the implication based distance measures with that of other measures from the literature, the distances defined in Eq.(9-12) have been participated in the comparative study presented herein.

In order to compare the examined distance measures, the performance index called *Degree of Confidence (DoC)* defined by the authors in [5] is also used. This factor measures the confidence of each distance metric in recognizing a specific sample that belongs to the pattern *(i)* and has the following form:

$$DoC^{(i)}=\sum_{i=1,i\neq j}^{n}\left|dist\left(P_{j},S\right)-dist\left(P_{i},S\right)\right| \tag{17}$$

It is obvious from the above Eq. (13) that the greater $DoC^{(i)}$ the more confident the result of the specific distance metric is.

The pattern classification problems used in this study and presented hereafter have been selected from the literature due to their popularity, since they used as benchmark datasets in evaluating the performance of fuzzy distance and/or similarity measures. It is worth noting that only the membership values of the attributes of those problems are used, since this data also includes hesitancy information, which is useful for intuitionistic fuzzy sets while this information is unnecessary in our case.

Finally, it is worth pointing out that in the following experiments, the proposed distance measure (d_{impD}) of Eq. (14) uses the D-implication

$$I_{D}\left(a,b\right)=\max\left\{\min\left\{1-a,1-b\right\},b\right\}.$$

4.1 Problem 1

This problem has been introduced in [17] and corresponds to a pattern classification problem of 4 classes and 12 attributes, described by the patterns P_1, P_2, P_3, P_4 and the test sample S, as presented in the following Table 1.

Table 1. 4-class/12-attributes problem [17], patterns and test sample

		Attributes											
		x_1	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**
Pattern #1	$\mu_{P_1}(x)$	0.173	0.102	0.530	0.965	0.420	0.008	0.331	1.000	0.215	0.432	0.750	0.432
Pattern #2	$\mu_{P_2}(x)$	0.510	0.627	1.000	0.125	0.026	0.732	0.556	0.650	1.000	0.145	0.047	0.760
Pattern #3	$\mu_{P_3}(x)$	0.495	0.603	0.987	0.073	0.037	0.690	0.147	0.213	0.501	1.000	0.324	0.045
Pattern #4	$\mu_{P_4}(x)$	1.000	1.000	0.857	0.734	0.021	0.076	0.152	0.113	0.489	1.000	0.386	0.028
Test sample	$\mu_S(x)$	0.978	0.980	0.798	0.693	0.051	0.123	0.152	0.113	0.494	0.987	0.376	0.012

For this example, it is prior known that the test sample belongs to class 4 and thus the distances have to take minimum values when the sample compared with the fourth pattern. Table 2, summarizes the distance measures' results along with the degree of confidence of each one. In this table the minimum distance and the two best distances with the highest degree of confidence, have been noted in bold.

Table 2. Distance measures' results

Distances	**Results**				
	dist(P_1,S)	**dist(P_2,S)**	**dist(P_3,S)**	**dist(P_4,S)**	$DoC^{(4)}$
d_H	5.401	5.591	2.460	**0.263**	**12.663**
$d_{n\text{-}H}$	0.450	0.466	0.205	**0.022**	1.055
d_E	1.799	1.778	1.064	**0.099**	4.345
$d_{n\text{-}E}$	0.519	0.513	0.307	**0.028**	1.254
d_L	3.668	3.467	2.520	**0.339**	8.638
d_M	3.447	3.760	2.431	**0.264**	8.847
d_{impD}	3.599	3.672	2.482	**0.282**	**8.907**

A careful study of the above table leads to the conclusion that while all the distances under comparison recognize correctly the test sample, the confidence of each distance measure varies. Although the *Hamming* distance *(d_H)* shows more confident than the other distances, the proposed D-implication distance (d_{impD}) is the second more confident measure by outperforming the distances that use the traditional *Mamdani* (d_M) and *Larsen* (d_L) implications.

4.2 Problem 2

This problem has been introduced in [18] and corresponds to a pattern classification problem of 3 classes and 3 attributes, described by the patterns P_1, P_2, P_3 and the test sample S, as presented in the following Table 3.

For this example, it is prior known that the test sample belongs to class 3. Table 4, summarizes the distance measures' results along with the degree of confidence of each one.

Table 3. 3-class/3-attributes problem [18], patterns and test sample

		Attributes		
		x_1	**x_2**	**x_3**
Pattern #1	$\mu_{P_1}(x)$	1.0	0.8	0.7
Pattern #2	$\mu_{P_2}(x)$	0.8	1.0	0.9
Pattern #3	$\mu_{P_3}(x)$	0.6	0.8	1.0
Test sample	$\mu_S(x)$	0.5	0.6	0.8

Table 4. Distance measures' results

Distances	**Results**			
	dist(P$_1$,S)	**dist(P$_2$,S)**	**dist(P$_3$,S)**	$DoC^{(3)}$
d_H	0.800	0.800	**0.500**	0.600
$d_{n\text{-}H}$	0.267	0.267	**0.167**	0.200
d_E	0.548	0.510	**0.300**	0.458
$d_{n\text{-}E}$	0.316	0.294	**0.173**	0.264
d_L	1.136	1.281	**0.755**	**0.908**
d_M	0.744	0.883	**0.456**	0.715
d_{impD}	0.949	0.883	**0.520**	**0.793**

The results are quite different as compared with the previous problem, since the most confident distance measure is that using the *Larsen* implication *(d_L)*, while the proposed measure remains the second most efficient one, by outperforming the rest distances.

4.3 Problem 3

This example has been introduced in [19,20] and corresponds to a pattern classification problem of 3 classes and 3 attributes, described by the patterns P$_1$, P$_2$, P$_3$ and the test sample S, as presented in the following Table 5.

Table 5. 3-class/3-attributes problem [19,20], patterns and test sample

		Attributes		
		x$_1$	**x$_2$**	**x$_3$**
Pattern #1	$\mu_{P_1}(x)$	0.1	0.5	0.1
Pattern #2	$\mu_{P_2}(x)$	0.5	0.7	0.0
Pattern #3	$\mu_{P_3}(x)$	0.7	0.1	0.4
Test sample	$\mu_S(x)$	0.4	0.6	0.0

For this example, it is prior known that the test sample belongs to class 2. Table 6, summarizes the distance measures' results along with the degree of confidence of each one.

The results in this problem are similar to those of the problem 1, with the *Hamming* distance being the dominant distance measure and the proposed metric being better than the *Larsen* implication based distance.

Conclusively, the proposed D-implication distance measure gives the same recognition rates with the other distances under comparison and in some cases is more confident than the traditional implications of *Mamdani* and *Larsen* and other distance measures from the literature.

Table 6. Distance measures' results

Distances	**Results**			
	dist(P$_1$,S)	**dist(P$_2$,S)**	**dist(P$_3$,S)**	$DoC^{(2)}$
d_H	0.500	**0.200**	1.200	**1.300**
$d_{n\text{-}H}$	0.167	**0.067**	0.400	0.433
d_E	0.332	**0.141**	0.707	0.756
$d_{n\text{-}E}$	0.191	**0.082**	0.408	0.436
d_L	0.326	**0.222**	0.552	0.435
d_M	0.546	**0.200**	0.766	**0.913**
d_{impD}	0.506	**0.221**	0.583	0.646

5 Conclusion

A novel distance measure between fuzzy sets based on fuzzy D-implications was proposed in the previous sections. The aforementioned distance shows a satisfactory behavior in classifying the patterns of a typical pattern classification problem, while it is high confident when compared to other implication based (*Mamdani, Larsen*) and traditional distance measures. It is worth noting that from the experimental study is concluded that each distance measure behaves differently in every problem and one has to evaluate several distances in order to decide which one is the most appropriate to use. In this context the proposed D-implication based distance measure constitutes an alternative choice, while the applied procedure that constructs it gives the flexibility to develop multiple distance measures of the same kind by using different type of D-implications.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Papakostas GA, Hatzimichailidis AG, Kaburlasos VG. Distance and similarity measures between intuitionistic fuzzy sets: A comparative analysis from a pattern recognition point of view. Pattern Recognition Letters. 2013;34(14):1609-1622.

[2] Chou CC. A new similarity measure of fuzzy numbers. Journal of Intelligent & Fuzzy Systems. 2014;26(1):287-294.

[3] Wei G, Lin R, Wang H. Distance and similarity measures for hesitant interval-valued fuzzy sets. Journal of Intelligent & Fuzzy Systems. 2014;27(1):19-36.

[4] Balopoulos V, Hatzimichailidis AG, Papadopoulos BK. Distance and similarity measures for fuzzy operators. Information Sciences. 2007;177(11):2336 -2348.

[5] Hatzimichailidis AG, Papakostas GA, Kaburlasos VG. A novel distance measure of intuitionistic fuzzy sets and its application to pattern recognition applications. International Journal of Intelligent Systems. 2012;27(4):396-409.

[6] Hatzimichailidis AG, Papakostas GA, Kaburlasos VG. A study on fuzzy D-implications. Proceedings of the 10th International FLINS Conference on Uncertainty Modeling in Knowledge Engineering and Decision Making (FLINS 2012), Istanbul, Turkey, 26-29 August. 2012;708-713.

[7] Hatzimichailidis AG, Papadopoulos BK. Similarity classes on fuzzy implications. Journal of Multiple-Valued Logic and Soft Computing. 2008;14(1-2):105-117.

[8] Klir G, Yuan B. Fuzzy sets and fuzzy logic theory and applications. Prentice Hall P T R Upper Saddle River, New Jersey, USA; 1995.

[9] Nguyen HT, Walker EA. A first course in fuzzy logic. CRC Press, Inc.; 1997.

[10] Xu Y, Ruan D, Qin K, Liu J. Lattice-valued logic. Springer-Verlag Berlin Heidelberg; 2003.

[11] Mas M, Monserrat M, Torrens J, Trillas E. QL-implications versus D-implications. Kybernetika. 2006;42(3):351-366.

[12] Baczynski M. Residual implications revisited. Notes on the Smerts-Magrez theorem. Fuzzy Sets and Systems. 2004;145(2):267-277.

[13] Fodor J. On contrapositive symmetry of implications in fuzzy logic. Proc. of the 1st European Congr. on Fuzzy and Intelligent Technologies, Aachen, Verlag der Augustinus Buchhandlung Aachen. 1993;1342–1348.

[14] Diamond P, Kloeden P. Metric spaces of fuzzy sets theory and applications. Word Scientific Publishing Co. Pte. Ltd, Singapore; 1994.

[15] Kacprzyk J. Multistage fuzzy control. Wiley, Chichester; 1997.

[16] Baker A. Matrix groups: An introduction to Lie group theory. Springer; 2001.

[17] Wang W, Xin X. Distance measure between intuitionistic fuzzy sets. Pattern Recognition Letters. 2005;26(13):2063-2069.

[18] Dengfeng L, Chuntian C. New similarity measures of intuitionistic fuzzy sets and application to pattern recognition. Pattern Recognition Letters. 2002;23(1-3):221-225.

[19] Hung WL, Yang MS. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern Recognition Letters. 2004;25(14):1603-1611.

[20] Hung WL, Yang MS. On the J-divergence of intuitionistic fuzzy sets with its applications to pattern recognition. Information Sciences. 2008;178(6):1641-1650.

Appendix

This Appendix presents the mathematical proofs of Propositions 1, 2 & 3 in section 2.2.

Proof of Proposition 1.

i) $(\Rightarrow)$

We have two different cases:

Case $x \le b$: From (i) it holds that $x \le b \Rightarrow I_D(a,x) \le I_D(a,b)$ (A1)

Moreover we have $I_D(a, x \wedge b) = I_D(a,x)$ and from (A1)

$\wedge\left(I_D(a,x), I_D(a,b)\right) = I_D(a,x)$. Therefore

$I_D(a, x \wedge b) = \wedge\left(I_D(a,x), I_D(a,b)\right)$ and property (i) has been proved.

Case $x \ge b$: From (i) it holds that $b \le x \Rightarrow I_D(a,b) \le I_D(a,x)$ (A2)

Moreover we have $I_D(a, x \wedge b) = I_D(a,b)$ and from (A2)

$\wedge\left(I_D(a,x), I_D(a,b)\right) = I_D(a,b)$. Therefore

$I_D(a, x \wedge b) = \wedge\left(I_D(a,x), I_D(a,b)\right)$ and property (i) has been proved.

$(\Leftarrow)$

We have two different cases:

Case $x \le b$: From (i) it holds that $I_D(a, x \wedge b) = \wedge\left(I_D(a,x), I_D(a,b)\right)$ (A3)

Moreover we have $I_D(a, x \wedge b) = I_D(a,x)$ and from (A3)

$I_D(a,x) = \wedge\left(I_D(a,x), I_D(a,b)\right)$

Therefore $I_D(a,x) \le I_D(a,b)$ and property (i) has been proved.

Case $x \ge b$: We have $I_D(a, x \wedge b) = I_D(a,b)$ and from (A3)

$I_D(a,b) = \wedge\left(I_D(a,x), I_D(a,b)\right)$.

Therefore $I_D(a,x) \ge I_D(a,b)$ and property (i) has been proved.

ii) This statement is proved in a similar way.

Proof of Proposition 2.

i) $(\Rightarrow)$

We have two different cases:

Case $a \le x$: From (i) it holds that $a \le x \Rightarrow I_D(a,b) \ge I_D(x,b)$ (A4)

Moreover we have $I_D(a \wedge x, b) = I_D(a,b)$ and from (A4)

$\vee(I_D(a,b), I_D(x,b)) = I_D(a,b)$. Therefore

$I_D(a \wedge x, b) = \vee(I_D(a,b), I_D(x,b))$ and property (i) has been proved.

Case $a \ge x$: From (i) it holds that $a \ge x \Rightarrow I_D(x,b) \ge I_D(a,b)$ (A5)

Moreover we have $I_D(a \wedge x, b) = I_D(x,b)$ and from (A5)

$\vee(I_D(a,b), I_D(x,b)) = I_D(x,b)$. Therefore

$I_D(a \wedge x, b) = \vee(I_D(a,b), I_D(x,b))$ and property (i) has been proved.

$(\Leftarrow)$

We have two different cases:

Case $a \le x$: From (i) it holds that $I_D(a \wedge x, b) = \vee(I_D(a,b), I_D(x,b))$ (A6)

Moreover we have $I_D(a \wedge x, b) = I_D(a,b)$ and from (A6)

$I_D(a,b) = \vee(I_D(a,b), I_D(x,b))$.

Therefore $I_D(a,b) \ge I_D(x,b)$ and property (i) has been proved.

Case $a \ge x$: We have $I_D(a \wedge x, b) = I_D(x,b)$ and from (A6)

$I_D(x,b) = \vee(I_D(a,b), I_D(x,b))$.

Therefore $I_D(x,b) \ge I_D(a,b)$ and property (i) has been proved.

ii) This statement is proved in a similar way.

Proof of Proposition 3.

We prove the property (i) and (v). The rest of the properties are proven in a similar way.

i) We have two different cases:

Case $a \le b$: It holds that $I_D(\wedge(a,b), c) = I_D(a,c) = \vee(\wedge(1-a, 1-c), c)$ (A7)

and $I_D(\vee(a,b), c) = I_D(b,c) = \vee(\wedge(1-b, 1-c), c)$ (A8)

From (A7) and (A8) we have $I_D(\wedge(a,b), c) \ge I_D(\vee(a,b), c)$.

Case $a \geq b$: The statement is proven in a similar way.

The property (i) has been proved.

v) From Proposition1 we have $a \leq b \Rightarrow I_D(a,x) \geq I_D(b,x)$. Therefore

$$\wedge\left(I_D(a,c), I_D(b,c)\right) = I_D(b,c) = S\left(T\left(n(b), n(c)\right), c\right) \leq S\left(T\left(n(a), n(b)\right), b\right) = I_D(a,b).$$

The property (v) has been proved.

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