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Analytical Approximate Solutions to the Nonlinear Singular Oscillator: An Iteration Procedure

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Authors' contributions

This work was carried out in collaboration between all authors. First author designed the study, wrote the protocol and supervised the work. First and second authors carried out all laboratories work and performed the statistical analysis. First author managed the analyses of the study. Second author wrote the first draft of the manuscript. Third and fourth authors edited the manuscript. First author managed the literature searches and edited the manuscript finally. All the authors read and approved the final manuscript.

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Abstract

Solution of the nonlinear singular oscillator has been obtained based on an iteration procedure. Here we have used a simple technique and taking a truncated Fourier series to determine the approximate analytic solution of the oscillator. The percentage of error between exact frequency and the third approximate frequency obtained by the adopted technique is as low as 0.696%. That is the third approximate frequency of the nonlinear singular oscillator shows a good agreement with its exact value. The convergent rate is high compared to other existing results. The modified technique introduces hopeful contrivance for many nonlinear oscillators.

Keywords: Iteration procedure; singular oscillator; nonlinearity; nonlinear oscillations.

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1 Introduction

In nature, most systems are inherently nonlinear. For this reason nonlinear equations are generally utilized in the Applied Mathematics, Physics and Engineering along with in other disciplines. That is why Engineers, Physicists, Mathematicians and most other Scientists are of interest to nonlinear problem. There are numerous methods to solve nonlinear problem like Perturbation method [1-6], Harmonic Balance (HB) method [7-11], Homotopy method [12-13], Iteration method [14-21] etc. The perturbation method is the most widely utilized method in which the nonlinear term is small. HB method is another latest method which is originated by Mickens [7] and farther work has been done by Lim [8], Gottlieb [9], Mickens [10] and so on for solving the strong nonlinear problems. Recently, some authors utilize an iteration procedure [14-21] which is valid for small together with large amplitude of oscillation, to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. A few number of scientist Hu and Tang [16]; Mickens [17]; Haque et al. [18-21] used modified version of this process to develop the results; luckily the direct iteration method sometimes improves the results when the functions are not differentiable that is for the singular oscillators. Beside this method, there are some methods [22-25] which are used to find approximate solution in the case of large amplitude of oscillations.

The simplicity and the excellent accuracy of the approximate solution of 'Nonlinear Singular Oscillator' by iteration procedure is the main object of the proposed procedure. Also, it can help us to investigate the nature (amplitude, frequency *etc*) in the nonlinear dynamical systems in a spacious range. Beside this technique, there are some articles [22-25] which show the excellent accuracy and stability of the oscillation method.

2 The Method

Let us suppose that a nonlinear oscillator modeled by

$$\ddot{x} + f(\ddot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0 \tag{1}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of Eq. (1), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, x) \equiv G(x, \ddot{x}). \tag{2}$$

Now, formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \; \mathrm{k} = 0, 1, 2, ... \tag{3}$$

Together with initial guess

$$x_0(t) = A\cos(\Omega_0 t) \tag{4}$$

Hence x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{5}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$. The above procedure gives the sequence of solutions which are mentioned by

$x_0(t),\ x_1(t),\ \cdots$ The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [14].

3 Solution Procedure

Let us consider the oscillator

$$\ddot{x} + x^{-1} = 0. \tag{6}$$

Adding $\Omega^2 x$ on both sides of Eq. (6), we get

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - x^{-1} \tag{7}$$

According to Eq. (3), the iteration scheme of Eq. (7) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^{-1} \tag{8}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting k=0 in Eq. (8) and using Eq. (4) we get

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - (A\cos\theta)^{-1} \tag{9}$$

Now expanding $(\cos\theta)^{-1}$ in a Fourier cosine series in the interval $[0, \frac{\pi}{2}]$ then Eq. (9) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = (\Omega_0^2 A - \frac{2}{A})\cos\theta + \frac{2}{A}\cos 3\theta - \cdots \tag{10}$$

Now secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$\text{i.e. } \Omega_0 = \frac{\sqrt{2}}{A} = \frac{1.41421}{A} \tag{11}$$

This is the first approximate frequency of the oscillator.

After simplification the Eq. (10) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \frac{2}{A}\cos 3\theta - \cdots \tag{12}$$

The complete solution is

$$x_1(t) = C\cos\theta - \frac{A}{8}\cos 3\theta - \cdots \tag{13}$$

Using $x_1(0) = A$, we have $C = \frac{9}{8}A$

Therefore, $x_1(t) = A(\frac{9}{8}\cos\theta - \frac{1}{8}\cos 3\theta + \cdots)$ (14)

This is the first approximate solution of the oscillator.

Proceeding to the second level of iteration, $x_2(t)$ satisfies the equation

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - (x_1)^{-1} \tag{15}$$

Now expanding second term on right hand side in a truncated Fourier cosine series in interval $[0, \frac{\pi}{2}]$ the Eq. (15) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\frac{9\Omega_1^2 A}{8} - \frac{2\sqrt{2}}{\sqrt{3}A})\cos\theta - (\frac{\Omega_1^2 A}{8} - \frac{1.30306}{A})\cos 3\theta + \cdots \tag{16}$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$\text{i.e. } \Omega_1 = \frac{1.2048}{A} \tag{17}$$

This is the second approximate frequency of the oscillator.

After simplification the Eq. (16) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = -(\frac{\Omega_1^2 A}{8} - \frac{1.30306}{A})\cos 3\theta + \cdots \tag{18}$$

The complete solution of Eq. (16) is

$$x_2(t) = C\cos\theta + (\frac{A}{64} - \frac{1.30306}{8A\Omega_1^2})\cos 3\theta - \cdots \tag{19}$$

Using $x_2(0) = A$, we have $C = (\frac{63}{64}A + \frac{1.30306}{8A\Omega_1^2})$

Therefore,

$$x_2(t) = 1.09659\, A\cos\theta - 0.0965883\, A\cos 3\theta + \cdots \tag{20}$$

Proceeding to the third level of iteration $x_3(t)$ satisfies the equation

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 x_2 - (x_2)^{-1} \tag{21}$$

Now expanding the term on right hand side in a Fourier cosine series in interval $[0, \frac{\pi}{2}]$, the Eq.(19) reduces to

$$\ddot{x}_3 + \Omega_2^2 x_3 = (1.09659 A\Omega_2^2 - \frac{1.69861}{A})\cos\theta + (-0.0965883 A\Omega_2^2 + \frac{1.42177}{A})\cos 3\theta - \cdots \quad (22)$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$\Omega_2 = \frac{1.24459}{A} \quad (23)$$

4 Results and Discussion

An iterative approach is presented to obtain approximate solution of the 'Nonlinear Singular Oscillator'. The present technique is very simple for solving algebraic equations analytically and the approach is different from the existing other approach for taking truncated Fourier series. Here we have calculated the first, second and third approximate frequencies Ω_0, Ω_1 and Ω_2 respectively. All the results are given in the following Table. To compare the approximate frequencies we have also given the existing results determined by Mickens iteration method [17], Mickens HB method [10] and Haque's iteration method [19]. To show the accuracy, we have calculated the percentage errors (denoted by Er(%)) by the definitions $\left|100(\Omega_e - \Omega_i)/\Omega_e\right|$, where $\Omega_i; i = 0, 1, 2, \cdots$ represents the approximate frequencies obtained by the present method and Ω_e represents the corresponding exact frequency of the oscillator.

Table. Comparison of the approximate frequencies with exact frequency Ω_e [17] of $\ddot{x} + x^{-1} = 0$

Exact frequency Ω_e	$\frac{1.253}{A}$		
Amplitude A	**First approximate frequency** Ω_0 **Er(%)**	**Second approximate frequency** Ω_1 **Er(%)**	**Third approximate frequency** Ω_2 **Er(%)**
Mickens iteration method [17]	$\frac{1.155}{A}$ 7.9	$\frac{1.018}{A}$ 18.1	–
Mickens Harmonic balance method [10]	$\frac{1.414}{A}$ 12.84	$\frac{1.273}{A}$ 1.6	$\frac{1.2731}{A}$ 1.58
Haque's iteration method [19]	$\frac{1.414}{A}$ 12.84	$\frac{1.208}{A}$ 3.63	$\frac{1.265}{A}$ 0.92
Adopted method	$\frac{1.414}{A}$ 12.84	$\frac{1.205}{A}$ 3.87	$\frac{1.245}{A}$ 0.696

In this article the result has been improved by rearranging the governing equation choosing comparatively perfect truncated Fourier series. In most of the articles, the results have been improved by modifying the method. But we see that not only modification of model is important but also rearranging of a nonlinear oscillator along with the Fourier series of initial solution of each iterative step is important in the case of iteration procedure.

5 Conclusion

Rearranging and applying truncated Fourier series is comparatively better. And the equation by an iteration technique from the first to the third approximate frequencies is better than corresponding frequencies which have been shown by other techniques. It can be observed that the Mickens' iteration technique is diverging here. Though the solution of Mickens' harmonic balance method and Haque's iteration method are convergent but the solution obtained by the adopted method is better than those mentioned solution. The third approximate frequency obtained by the adopted technique is as low as 0.696%. Thus we can say third approximation provides excellent result. Furthermore, this technique is precious because it does not require numerical integration for expanding the function in Fourier series in any step of iterations. Thus we conclude that this technique does not only presents explicitly a better third order analytical solutions but also deduces a simply way of starting fourth, fifth etc order solutions of various nonlinear systems.

Competing Interests

Authors have declared that no competing interests exist.

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