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Analysis in the Generalized Function Spaces $L_\alpha(S_\alpha^\alpha(R))$

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

In this paper we shall define generalized complex numbers in the space of new generalized functions $L_\alpha(S_\alpha^\alpha(R))$, which allows us to define and study ordinary differential equations (ODE) and partial differential equations (PDE) in the space $L_\alpha(S_\alpha^\alpha(R))$. We shall also define generalized linear and bilinear operations in the space $L_\alpha(S_\alpha^\alpha(R))$.

Keywords: New generalized function; generalized functions.

1. INTRODUCTION

The space of new generalized functions $L_\alpha(S_\alpha^\alpha(R))$ is defined [1] as a factor algebra $L_\alpha(S_\alpha^\alpha(R)) = G_\alpha(S_\alpha^\alpha(R)) / N_\alpha(S_\alpha^\alpha(R))$, where $S_\alpha^\alpha(R)$ be Rome- Shilov – Gelfand space (see [2]), and $G_\alpha(S_\alpha^\alpha(R)), N_\alpha(S_\alpha^\alpha(R))$ be a subsets of $G(S_\alpha^\alpha(R))$ - the set of all possible sequences in $S_\alpha^\alpha(R)$.

The following results shows the importance of the space of new generalized functions $L_\alpha(S_\alpha^\alpha(R))$:

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Theorem 1.1. The space $G_\alpha(S_\alpha^\alpha(R))$ is a Sub algebra of the algebra $G(S_\alpha^\alpha(R))$, and $N_\alpha(S_\alpha^\alpha(R))$ is an ideal of $G(S_\alpha^\alpha(R))$.

The topology in $S_\alpha^\alpha(R)$ is defined by the system of semi norms in the following way:

$$p_{n,l} = \sup_{\substack{k\le n \\ m\le l}} q_{k,m}(t) \quad \text{where}$$

$$q_{m,k} = \sup_{x\in R} \frac{x^k f^{(m)}(x)}{A^k B^m k^{\alpha k} m^{\alpha m}}$$

Theorem 1.2. If $f(x), g(x) \in S_\alpha^\alpha(R)$, then for each n,l there is a constant $C_{n,l} > 0$ such that $p_{n,l}(fg) \le C_{n,l}\, p_{n,l}(f)\, p_{n,l}(g) \quad \forall f, g \in S_\alpha^\alpha(R)$. (see [1,3-7]).

Theorem 1.3. The following embeddings are true [8]:

$S_\alpha^\alpha(R) \subset L_\alpha(S_\alpha^\alpha(R))$ and $[S_\alpha^\alpha(R)]^* \subset L_\alpha(S_\alpha^\alpha(R))$, where $[S_\alpha^\alpha(R)]^*$ be the dual of the space $S_\alpha^\alpha(R)$

Now we can define the associative multiplication for element of the space $[S_\alpha^\alpha(R)]^*$ as elements of the algebra $L_\alpha(S_\alpha^\alpha(R))$.

There arise a natural question; is it possible to define linear and bilinear operations on the algebra $L_\alpha(S_\alpha^\alpha(R))$ and is it possible to define tools in this space such that the ordinary and partial differential equations will be study and correct define ?

In this paper we will define linear and bilinear operations in the space of new generalized function space $L_\alpha(S_\alpha^\alpha(R))$. Also we will extend lebesgue's integral in the algebra $L_\alpha(S_\alpha^\alpha(R))$.

2. ANALYSIS IN THE SPACE $L_\alpha(S_\alpha^\alpha(R))$

In order to define the differential equations with initial conditions in the space of new generalized functions $L_\alpha(S_\alpha^\alpha(R))$ in [8] we defined set of generalized complex numbers in the following way:

Suppose $G(C) = \{(z_k)_1^\infty : z_k \in C\}$ be the set of all complex sequences.

Define the following subsets of the set $G(C)$:

$$G_M(C) = \left\{ (z_k)_1^\infty \in G(C) : \exists m \in N, \exists c \in R \;\; \forall k \quad |z_k| < c\, e^{mk^{\frac{1}{\alpha}}} \right\}$$

$$N(C) = \left\{ (z_k)_1^\infty \in G(C) : \forall k, \forall m \in N, \exists c \in R \; : |z_k| < c\, e^{-mk^{\frac{1}{\alpha}}} \right\}$$

Theorem 2.1. Let $f = (f_k) \in G_\alpha(S_\alpha^\alpha(R))$, and $g = (g_k) \in N_\alpha(S_\alpha^\alpha(R))$, and suppose $z_0 \in C$, then $f(z_0) = (f_k(z_0)) \in G_M(C)$, and $g(z_0) = (g_k(z_0)) \in N(C)$.

Proof. The proof implies directly from the definitions and properties of the sets $G_\alpha(S_\alpha^\alpha(R)), N_\alpha(S_\alpha^\alpha(R)), G_M(C), N(C)$.

Theorem 2.2. The set $G_M(C)$ is a sub algebra of algebra $G(C)$, and the set $N(C)$ is an ideal in the algebra $G_M(C)$.

Proof. Suppose $z = (z_k)$, $\eta = (\eta_k)$ are elements in $G_M(C)$, then there are natural numbers m_1, m_2 and a constants $c_1 > o$, $c_2 > 0$ such that

$$|z_k| \le C_1 \exp(m_1 k^{\frac{1}{\alpha}}), \quad |\eta_k| \le C_2 \exp(m_2 k^{\frac{1}{\alpha}})$$

since $|z_k \eta_k| \le C_1 C_2 \exp((m_1 + m_2) k^{\frac{1}{\alpha}})$, then $z\eta \in G_M(C)$ which means that $G_M(C)$ is a sub algebra of algebra $G(C)$.

Now suppose that $z = (z_k) \in G_M(C)$, and $w = (w_k) \in N(C)$ Since $|z_k w_k| \le Cd \exp((m_1 - m) k^{\frac{1}{\alpha}})$, then $zw \in N(C)$ which means that $N(C)$ is an ideal in the algebra $G_M(C)$.

Now we define the generalized numbers in the following way:

The elements $z = (z_k) \in G_M(C),\ \eta = (\eta_k) \in G_M(C)$ are called equivalence ($z \sim \eta$) If and only if $(z_k - \eta_k) \in N(C)$.

The set of all equivalence classes $\tilde{C} = G_M(C)/N(C)$ is called the set of generalized numbers corresponding to the space $L_\alpha(S_\alpha^\alpha(R))$. It is clear that $C \subset \tilde{C}$.

The set of generalized functions allow us to define and study many mathematical models in our space, for example the differential equation

$$P(D)u = f,\ u(0) = b\ ,$$

where

$$P(D)u = a_n u^{(n)} + a_{n-1} u^{(n-1)} + ... + a_1 u' + a_0 u\ ,$$

$$a_i \in C, a_n \ne 0\ ;\ u, f \in L_\alpha(S_\alpha^\alpha(R)), b \in \tilde{C}$$

We can solve by applying the Fourier transform F to both sides to get the algebraic system

$$P(ix)F(u_k) = F(f_k)\ ,\ u(0) = b_k\ .$$

Also we can define and study the Lebesgue's integral in our space $L_\alpha(S_\alpha^\alpha(R))$:

let $f \in L_\alpha(S_\alpha^\alpha(R))$, and let K be a compact set in R. Now let (f_k) be any representative of f, then

$$\left|\int_K f(x)\,dx\right| = \left|\int_K f_k(x)\,dx\right| \le \int_K |f_k(x)|\,dx \le c\, e^{mk^{\frac{1}{\alpha}}}$$

which means $\int_K f(x)\,dx \in G_M(C)$. Now if (f_k^*) be other representative of the generalized function f, then

$$\left|\int_K f_k(x)\,dx - \int_K f_k^*(x)\,dx\right| = \left|\int_K (f_k(x) - f_k^*(x))\,dx\right| \le c\, e^{-mk^{\frac{1}{\alpha}}}$$

Which means that the Lebesgue's integral is independent on representative of f.

Remark. It is easy to check that the extended Lebesgue's Integral defined above in the space $L_\alpha(S_\alpha^\alpha(R))$ satisfies many properties of usual one, for example the linearity:

$$\int_K [\alpha f(x) + \beta g(x)]\,dx = \alpha \int_K \alpha f(x)\,dx + \beta \int_K g(x)\,dx\ ,\ \forall f, g \in L_\alpha(S_\alpha^\alpha(R))\ ,\ \alpha, \beta \in \tilde{C}$$

Similarly we can extend any linear continuous functional

$$< f, \varphi >:\ S_\alpha^\alpha(R) \to C$$

to

$$< f^*, \varphi >:\ L_\alpha(S_\alpha^\alpha(R)) \to \tilde{C}$$

since $f^*[G_\alpha(S_\alpha^\alpha(R))] \subset G_\alpha(S_\alpha^\alpha(R))$, and $f^*[N_\alpha(S_\alpha^\alpha(R))] \subset N_\alpha(S_\alpha^\alpha(R))$.

3. LINEAR AND BILINEAR CONTINUOUS OPERATORS IN $L_\alpha(S_\alpha^\alpha(R))$

Now let $A: S_\alpha^\alpha(R) \to S_\alpha^\alpha(R)$ be a linear continuous operator and let $B: S_\alpha^\alpha(R) x S_\alpha^\alpha(R) \to S_\alpha^\alpha(R)$ be a bilinear continuous operation, then [9] for each $i \in I$ there exist $j \in I$ and a constant $c > 0$ such that

$P_i(A(u(x))) \le c_i \; P_j(u(x))$, $\forall u \in S_\alpha^\alpha(R)$, and for each $\alpha \in I$ there is $\beta \in I$ and constant $c > 0$ such that

$$P_\alpha(B(f,g) \le c \;\; P_\beta(f) P_\beta(g) \quad \forall f, g \in S_\alpha^\alpha(R)$$

We extend A, and B in the space $B(G_\alpha(S_\alpha^\alpha(R))$ by the following:

If $f = (f_k) \in \; G_\alpha(S_\alpha^\alpha(R))$, and $g = (f_k) \in \; G_\alpha(S_\alpha^\alpha(R))$, then define

$A^*(f) = (A(f_k))$, and $B^*(f,g) = (B(f_k), B(g_k))$

Theorem 3.1. Let $A: S_\alpha^\alpha(R) \to S_\alpha^\alpha(R)$ be a linear continuous operator and let $B: S_\alpha^\alpha(R) x S_\alpha^\alpha(R) \to S_\alpha^\alpha(R)$ be a bilinear continuous operation, then:

a) $B(G_\alpha(S_\alpha^\alpha(R)), G_\alpha(S_\alpha^\alpha(R))) \subset G_\alpha(S_\alpha^\alpha(R))$; $B[N_\alpha(S_\alpha^\alpha(R), N_\alpha(S_\alpha^\alpha(R))] \subset N_\alpha(S_\alpha^\alpha(R))$

b) $A\; [G_\alpha(S_\alpha^\alpha(R))] \subset G_\alpha(S_\alpha^\alpha(R))$, and $A[N_\alpha(S_\alpha^\alpha(R))] \subset N_\alpha(S_\alpha^\alpha(R))$;

c) The operator A_α^* is independent on a representative;

d) The bilinear operation B_α^* on representatives.

Proof. The proof of parts a and b follows immediately from the definitions and properties of the sets $G_\alpha(S_\alpha^\alpha(R))$, and $N_\alpha(S_\alpha^\alpha(R))$.

c) suppose $f \in L_\alpha(S_\alpha^\alpha(R))$, and let (f_k) and (g_k) be are two representative of f.

Consider $P_i(A_\alpha^*(f_k) - A_\alpha^*(g_k)) = P_i(A_\alpha^*(f_k - g_k)) \le c_j \;\; P_j\left(f_k - g_k\right) \le c\; e^{-mk^{\frac{1}{\alpha}}}$ which means that $(A_\alpha^*(f_k) - A_\alpha^*(g_k)) \in \; N_\alpha(S_\alpha^\alpha(R))$

d) let (f_k) and (g_k) be are representatives of $f, g \in L_\alpha(S_\alpha^\alpha(R))$, and let (f_k^*) and (g_k^*) be are other representatives consider

$$P_i(B(f_k, g_k) - B(f_k^*, g_k^*)) \le P_i\;(B(f_k, g_k) - B(f_k^*, g_k)) + P_i(B(f_k, g_k) - B(f_k^*, g_k^*)) =$$

$$P_i(B(f_k - f_k^*, g_k)) + P_i(B(f_k^*, g_k - g_k^*)) \le c_0 \;\; P_\beta(f_k - f_k^*) P_\beta(g_k) + c_0 P_\beta(f_k^*) P_\beta(g_k - g_k^*) \le ce^{-mk^{\frac{1}{\alpha}}}$$

that is $(B(f_k, g_k) - B(f_k^*, g_k^*)) \in \; N_\alpha(S_\alpha^\alpha(R))$.

Now we can lift A^*, and B^* to our space $L_\alpha(S_\alpha^\alpha(R))$

$$A_\alpha^* : L_\alpha(S_\alpha^\alpha(R)) \to L_\alpha(S_\alpha^\alpha(R))$$

and

$$B_\alpha^* : L_\alpha(S_\alpha^\alpha(R)) x L_\alpha(S_\alpha^\alpha(R)) \to L_\alpha(S_\alpha^\alpha(R))$$

4. CONCLUSION

In this paper, we have defined generalized complex numbers and generalized linear and bilinear continuous operators to define ordinary differential equations and partial differential equations in the new generalized function space $L_\alpha(S_\alpha^\alpha(R))$.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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