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# A Coincidence Point Theorem in Ordered Cone Metric Spaces

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***Author's contribution***

*The sole author designed, analyzed and interpreted and prepared the manuscript.*

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## ABSTRACT

In this paper, we obtain a coincidence point result in cone metric spaces, without appealing the normality condition. These results extend and improve some known results.

*Keywords: Cone metric space; coincidence point; compatible maps; weakly decreasing type A.*

## 1. INTRODUCTION AND PRELIMINARIES

Generalizing the usual metric space by replacing the set of real numbers by an ordered Banach space and introduced the cone metric space by Huang and Zhang [1] and obtained some fixed point theorems in this cone metric space of different types of contractive mappings and using the normal cone. Since then, many authors have studied Huang and Zhang [1] fixed point theorems and extended the fixed point theorems of Huang and Zhang [1] in cone metric spaces using, with and without the normal cone (see, e.g., [2-9]). Very recently, W. Shatanawi [10] introduced the concept of a weakly decreasing type A and proved some results on coincidence

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point theorems in cone metric spaces, without appealing the normality. In this paper, we obtain a coincidence point theorem without using the normality condition in an ordered cone metric spaces, which is an extension of the Theorem 2.2 of [10].

Let E be a real Banach space and P be a subset is called a cone iff

(i) P is a closed and non-empty set and $P \neq \{\theta\}$,

(ii). $a,b \epsilon R$ with $a,b \geq 0$, $x,y \in P$ implies $ax+by \epsilon P$,

(ii). $P \cap (-P)=\{\theta\}$.

For a cone P, and define a partial ordering ≼ with respect to P by $x \preccurlyeq y$ iff $y-x \epsilon P$. We write $x \prec y$ to indicate $x \preccurlyeq y$ but $x \neq y$ , while $x << y$ will stand for $y-x \in \text{int } P$ , where int P is the interior of P. This cone P is called an order cone.

The order cone P is said to be a normal cone if there exists a M > 0 such that

$$\theta \preccurlyeq x \preccurlyeq y \Longrightarrow \|x\| \leq M\|y\| \text{ for all } x,y \epsilon E.$$

The least positive integer M is called a normal constant.

### 1.1 Definition

[1] Let X be a nonempty set. And suppose a mapping $d: X \times X \to E$ satisfy the following

(i) $\theta \prec d(x,y)$ and $d(x,y) = \theta$ iff $x = y$, for all $x,y \epsilon X$,

(ii) $d(x, y) = d(y, x)$ for all $x,y \epsilon X$,

(iii) $d(x,y) \preccurlyeq d(x,z)+d(y,z)$ for all $x,y,z \epsilon X$.

Then (X, d) is a cone metric space, where d is called a cone metric in X.

### 1.2 Definition

[1] Let $\{x_n\}$ be a sequence in a cone metric space (X,d), $x \epsilon X$. If for every $c \epsilon E$ with $c >> \theta$, and $K \epsilon Z^+$ such that $d(x_n, x_m) << c$ for all $n,m>K$, then the sequence $\{x_n\}$ is called a Cauchy sequence.

### 1.3 Definition

[1] Let $\{x_n\}$ be a sequence in a cone metric space (X, d) and $x \epsilon X$. If for any $c >> \theta$, and $K \epsilon Z^+$ such that $d(x_n ,x ) << c$ for all $n>K$ , then the sequence $\{x_n\}$ is said to be a convergent. And we denote this convergent sequence, $x_n \to x$ as $n \to \infty$.

The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent.

### 1.4 Definition

[10] Let f, T be two self-mappings and $(X, \sqsubseteq)$ be a partially ordered set. If

(a) $fx \sqsubseteq fy$, for all $x \epsilon X$, for all $y \epsilon T^{-1}(fx)$.

(b) $TX \subseteq fX$.

Then f is weakly decreasing type A with respect to T.

### 1.5 Definition

[11] let f, g be two self mappings in a cone metric space (X, d). Then the pair (f, g) is said to be compatible if for an arbitrary sequence $\{x_n\}$ $\subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \epsilon X$, and for arbitrary $c \epsilon$ int P, there exists $m \epsilon N$ such that $d(fgx_n, gfx_n) << c$ , whenever $n > m$. And compatible is said to be weakly compatible if $fx = gx \Longrightarrow fgx = gfx$.

### 1.6 Definition

[2] let f and g be two self mappings of a set X. If $q = fp = gp$ for some point p in X, then p is called a coincidence point of f and g and q is called a point of coincidence of f and g .

## 2. MAIN RESULTS

In this section, we obtain a coincidence point theorem for mappings in cone metric spaces without assuming the normal condition.

### 2.1 Theorem

Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a solid cone P. Let $f,T:X \to X$ be two maps such that. Assume that f is a weakly decreasing type A with respect to T and f,T are satisfy the following :

$$d(Tx,Ty) \preccurlyeq p\, d(fx,fy)+ q\, d(fx,Tx)+ r\, d(fy,Ty) +s\, d(fx,Ty)+t\, d(fy,Tx) \quad (2.1)$$

for all $x,y \epsilon X$ for which fx and fy are comparable, where p, q, r, s and t are non negative real numbers with $p+q+r+s+t \in [0,1)$.

And also assume that {f, T} is compatible and f, T are continuous .Then f and T have a coincidence point in X, that is there exists a point u in X such that fu = Tu.

### 2.1.1 Proof

Let $x_0$ be an arbitrary point in X, and there exists $x_1 \epsilon X$ such that $Tx_0 = fx_1$, since $TX \subseteq fX$. Similarly, $x_2 \epsilon X$ such that such that $Tx_1 = fx_2$.Continuining this process, and construct a sequence $\{x_n\}$ in X such that $Tx_n = fx_{n+1}$. Since, $x_n \epsilon T^{-1}(fx_{n+1})$, $n \epsilon N$, that is, f is a weakly decreasing of type A with respect to T, we have

$$fx_0 \sqsupseteq fx_1 \sqsupseteq fx_2 \sqsupseteq fx_3 \sqsupseteq \ldots$$

We have, by the (2.1)

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\preccurlyeq p\, d(fx_n, fx_{n+1}) + q\, d(fx_n, Tx_n) + rd(fx_{n+1}, Tx_{n+1}) + s\, d(fx_n, Tx_{n+1}) + t\, d(fx_{n+1}, Tx_n) \\ &\preccurlyeq p\, d(Tx_{n-1}, Tx_n) + q\, d(Tx_{n-1}, Tx_n) + rd(Tx_n, Tx_{n+1}) + s\, d(Tx_{n-1}, Tx_{n+1}) + t\, d(Tx_n, Tx_n) \\ &\preccurlyeq p\, d(Tx_{n-1}, Tx_n) + q\, d(Tx_{n-1}, Tx_n) + rd(Tx_n, Tx_{n+1}) + s\, d(Tx_{n-1}, Tx_{n+1}) \\ &\preccurlyeq p\, d(Tx_{n-1}, Tx_n) + q\, d(Tx_{n-1}, Tx_n) + rd(Tx_n, Tx_{n+1}) + s\, [d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) \\ &\preccurlyeq (p+q+s) d(Tx_{n-1}, Tx_n) + (r+s) d(Tx_n, Tx_{n+1}). \end{aligned} \quad (2.2)$$

And

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\preccurlyeq pd(fx_{n+1}, fx_n) + qd(fx_{n+1}, Tx_{n+1}) + rd(fx_n, Tx_n) + sd(fx_{n+1}, Tx_n) + td(fx_n, Tx_{n+1}), \\ &\preccurlyeq pd(Tx_n, Tx_{n-1}) + qd(Tx_n, Tx_{n+1}) + rd(Tx_{n-1}, Tx_n) + sd(Tx_n, Tx_n) + td(Tx_{n-1}, Tx_{n+1}), \\ &\preccurlyeq pd(Tx_n, Tx_{n-1}) + qd(Tx_n, Tx_{n+1}) + rd(Tx_{n-1}, Tx_n) + t\,[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})], \\ &\preccurlyeq (p+r+t)\, d(Tx_n, Tx_{n-1}) + (q+t)\, d(Tx_n, Tx_{n+1}). \end{aligned} \quad (2.3)$$

Now,

$$\begin{aligned} 2d(Tx_{n+1}, Tx_n) &= d(Tx_{n+1}, Tx_n) + d(Tx_n, Tx_{n+1}) \\ &\preccurlyeq (p+r+t) d(Tx_n, Tx_{n-1}) + (q+t)\, d(Tx_n, Tx_{n+1}) + (p+q+s)\, d(Tx_{n-1}, Tx_n) + (r+s)\, d(Tx_n, Tx_{n+1}), \end{aligned}$$

(By the 2.2 and 2.3)

$$\preccurlyeq (2p+q+r+s+t)\, d(Tx_{n-1}, Tx_n) + (q+r+s+t) d(Tx_n, Tx_{n+1}).$$

$$(2-q-r\,s-t) d(Tx_{n+1}, Tx_n) \preccurlyeq (2p+q+r+s+t)\, d(Tx_{n-1}, Tx_n),$$

$$d(Tx_{n+1}, Tx_n) \preccurlyeq (2p+q+r+s+t)/(2-q-r-s-t) d(Tx_{n-1}, Tx_n)$$

Put, $b = (2p+q+r+s+t)/(2-q-r-s-t) < 1$.

Then we obtain,

$$d(Tx_n, Tx_{n+1}) \preccurlyeq b\, d(Tx_{n-1}, Tx_n). \quad (2.4)$$

Thus, for $n \epsilon N$, we have

$$d(Tx_n, Tx_{n+1}) \preccurlyeq b\, d(Tx_{n-1}, Tx_n) \preccurlyeq b^2 d(Tx_{n-2}, Tx_{n-1}) \preccurlyeq \ldots \preccurlyeq b^n d(Tx_0, Tx_1).$$

Let, $n, m \epsilon N$ with $m > n$. Then

$$d(Tx_n, Tx_m) \preccurlyeq \sum_{i=n}^{m-1} b^i\, d(Tx_i, Tx_{i+1}),$$

$$\preccurlyeq \sum_{i=n}^{m-1} b^i\, d(Tx_0, Tx_1).$$

Since, $b \epsilon [0,1)$, we have

$$d(Tx_n, Tx_m) \preccurlyeq b^n /1-b\, d(Tx_0, Tx_1) \to \theta \text{ as } n \to \infty \quad (2.5)$$

Claim: $\{Tx_n\}$ is a Cauchy sequence in X.

Let $\theta << e$ be given.

Since, $e \epsilon$ Int P, then there exists a neighborhood

$$N_\delta(\theta) = \{x \in E : \|x\| < \delta\},\ \delta > 0, \text{ such that } e + N_\delta(\theta) \subseteq \text{Int } P.$$

Also, choose a natural number $N_1$ such that $b^n /1-b\, d(Tx_0, Tx_1) \epsilon N_\delta(\theta)$ for all $n \geq N_1$.

Hence, $e - b^n /1-b\ \epsilon\ e + N_\delta(\theta) \subseteq$ Int P.
Thus, we have that for all $n \geq N_1$,

$b^n$ /1-b d ($Tx_0$, $Tx_1$) <<e. (2.6)

From (2.5) and (2.6), we get that
d($Tx_n$, $Tx_m$)<<e , whenever n≥$N_1$.

Therefore, {$Tx_n$} is a Cauchy sequence in X. Hence, the claim. Since X is a complete, then there is a point

uϵX such that $Tx_n$ →u(as, n →∞).

From the continuity of f and T, we have

T($Tx_n$) →Tu as n→ ∞ and f ($Tx_n$)→fu as n→ ∞.

From the triangle inequality, we get that

d(Tu,fu)≼d(Tu,T($Tx_n$))+d(T($Tx_n$),f($Tx_{n+1}$))
+d(f($Tx_{n+1}$),fu),
= d(Tu, T($Tx_n$))+d(T($fx_{n+1}$),f($Tx_{n+1}$))
+d(f($Tx_{n+1}$),fu). (2.7)

Let θ<<e be given. Then,

there exists $k_1$ = $k_1$ (e ) such that

d(Tu, T($Tx_n$)) << e/3 for all n≥$k_1$. (2.8)

Note that, $fx_{n+1}$ = $Tx_n$ →u, as n→∞
and $Tx_{n+1}$ →u, as n→∞.

Since, {T, f} is compatible, then

there exists $k_2$ = $k_2$( e ) such that

d(T($fx_{n+1}$), f($Tx_{n+1}$)) <<e/3 for all n ≥$k_2$. (2.9)

Now, there is $k_3$ = $k_3$( e ) such that

d(T($fx_{n+1}$), fu) << e/3 for all n ≥ $k_3$. (2.10)

Let $k_0$ = max {$k_1$, $k_2$, $k_3$}.

From (2.7), (2.8), (2.9) , (2.10), we obtain that

d(Tu, fu) <<e/3+e/3+e/3 =e.

Since, e is arbitrary, we conclude that

d(Tu, fu) << e/r for each rϵN. Noting that e/r→ θ as r→∞, we have that

e/r- d(Tu, fu) → - d(Tu, fu) as r→∞.

Since P is closed, - d (Tu, fu)ϵ P.
Implies, d(Tu, fu)ϵ P∩ (-P).

Implies, d(Tu, fu) = θ.

Therefore, f and T have a coincidence point uϵX.

### 2.2 Remark

If we can choose, s = t = 0 in the above Theorem 2.1, then we obtain the Theorem 2.2 of [10].

### 2.3 Remark

If we can choose p = k and q = r = s = t = 0 in the above Theorem 2.1, then we obtain the following corollary.

### 2.4 Corollary

Let (X, d) be a complete cone metric space and(X,⊑) be partially ordered set, P is a solid cone. Let T, f: X→X be two mappings. Assume that f is weakly decreasing type A with respect to T, and f,T satisfy the following:

d(Tx,Ty) ≼ kd(fx, fy) (2.11)

for all x, yϵX for which fx and fy are comparable, where k≥0 with kϵ [0,1) . And assume that the pair {T, f} is compatible T and T, f are continuous. Then T and f have a coincidence point in X.

## 3. CONCLUSION

In cone metric spaces, we can get the fixed points without appealing the normality, which are more general results than the previous results of [1].

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## COMPETING INTERESTS

Author has declared that no competing interests exist.

## REFERENCES

1. Huang LG, Zhang X. Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. 2007; 332(2):1468-1476.
2. Abbas M, Jungck G. Common fixed point results for non commuting mappings

without continuity in cone metric spaces. J. Math. Anal. Appl. 2008;341:416-420.
3. Altun I, Durmaz G. Some fixed point theorems on ordered cone metric spaces. Rend. Ciric. Math. Palermo. 2009;5(8):319-325.
4. Arshad M, Azam A, Vetro P. Some common fixed point results on cone metric spaces. Fixed Point Theory Appl. 2009;11. Article ID 493965.
5. Azam A, Arshad M, Beg I. Common fixed points of two maps in cone metric spaces, Rend. Circ. Mat. Palermo. 2008;57:433-441.
6. Beg I, Abbas M. Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition. Fixed Point Theory Appl. 2006(2006): 3-7. Article ID 74503.
7. Choudhry BS, Metiya N. The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Comput. Math. Appl. 2010;60(6):1686-1695.
8. Vetro P. Common fixed points in cone metric spaces. Rend. Cric. Mat. Palermo. 2010;56:464-468.
9. Rezapur S, Hamlbarani. Some notes on the paper cone metric spaces and fixed point theorem of contractive mappings. J. Math. Anal. Appl. 2008;345:719-724.
10. Shatanawi W. Some coincidence point results in cone metric spaces. Math. Comp. Modelling. 2012;55(7-8):2023-2028.
11. Jankovic S, Golubovic Z, Radenovic S. Compatible and weakly compatible mappings in cone metric spaces. Math. Comput. Modelling. 2010;52:1728-1738.



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